

Appendix I

Hints to Starred Exercises

1.2.9(b) If $G \neq T_{m,n}$, then G has parts of size n_1, n_2, \dots, n_m , with $n_i - n_j > 1$ for some i and j . Show that the complete m -partite graph with parts of size $n_1, n_2, \dots, n_i - 1, \dots, n_j + 1, \dots, n_m$ has more edges than G .

1.3.3 In terms of the adjacency matrix \mathbf{A} , an automorphism of G is a permutation matrix \mathbf{P} such that $\mathbf{PAP}' = \mathbf{A}$ or, equivalently, $\mathbf{PA} = \mathbf{AP}$ (since $\mathbf{P}' = \mathbf{P}^{-1}$). Show that if \mathbf{x} is an eigenvector of \mathbf{A} belonging to an eigenvalue λ , then, for any automorphism \mathbf{P} of G , so is \mathbf{Px} . Since the eigenvalues of \mathbf{A} are distinct and \mathbf{P} is orthogonal, $\mathbf{P}^2\mathbf{x} = \mathbf{x}$ for all eigenvectors \mathbf{x} .

1.4.5 Suppose that all induced subgraphs of G on n vertices have m edges. Show that, for any two vertices v_i and v_j ,

$$\varepsilon(G) - d(v_i) = \varepsilon(G - v_i) = m \binom{\nu - 1}{n} / \binom{\nu - 3}{n - 2}$$

$$\varepsilon(G) - d(v_i) - d(v_j) + a_{ij} = \varepsilon(G - v_i - v_j) = m \binom{\nu - 2}{n} / \binom{\nu - 4}{n - 2}$$

where $a_{ij} = 1$ or 0 according as v_i and v_j are adjacent or not. Deduce that a_{ij} is independent of i and j .

1.5.7(a) To prove the necessity, first show that if G is simple with $u_1v_1, u_2v_2 \in E$ and $u_1v_2, u_2v_1 \notin E$, then $G - \{u_1v_1, u_2v_2\} + \{u_1v_2, u_2v_1\}$ has the same degree sequence as G . Using this, show that if \mathbf{d} is graphic, then there is a simple graph G with $V = \{v_1, v_2, \dots, v_n\}$ such that (i) $d(v_i) = d_i$ for $1 \leq i \leq n$, and (ii) v_1 is joined to $v_2, v_3, \dots, v_{d_1+1}$. The graph $G - v_1$ has degree sequence \mathbf{d}' .

1.5.8 Show that a bipartite subgraph with the largest possible number of edges has this property.

1.5.9 Define a graph on S in which x_i and x_j are adjacent if and only if they are at distance one. Show that in this graph each vertex has degree at most six.

1.7.3 Consider a longest path and the vertices adjacent to the origin of this path.

1.7.6(b) By contradiction. Let G be a smallest counter-example. Show that (i) the girth of G is at least five, and (ii) $\delta \geq 3$. Deduce that $\nu \leq 8$ and show that no such graph exists.

2.1.10 To prove the sufficiency, consider a graph G with degree sequence $\mathbf{d} = (d_1, d_2, \dots, d_\nu)$ and as few components as possible. If

G is not connected, show that, by a suitable exchange of edges (as in the hint to exercise 1.5.7a), there is a graph with degree sequence \mathbf{d} and fewer components than G .

2.2.12 Define a labelled graph G as follows: the vertices of G are the subsets A_1, A_2, \dots, A_n , and A_i is joined to A_j ($i \neq j$) by an edge labelled a if either $A_i = A_j \cup \{a\}$ or $A_j = A_i \cup \{a\}$. For any subgraph H of G , let $L(H)$ be the set of labels on edges of H . Show that if F is a maximal forest of G , then $L(F) = L(G)$. Any element x in $S \setminus L(F)$ has the required property.

2.4.2 Several applications of theorem 2.8 yield the recurrence relation

$$w_n - 4w_{n-1} + 4w_{n-2} - 1 = 0$$

where w_n is the number of spanning trees in the wheel with n spokes. Solve this recurrence relation.

3.2.6 Form a new graph G' by adding two vertices x and y , and joining x to all vertices in X and y to all vertices in Y . Show that G' is 2-connected and apply theorem 3.2.

3.2.7(a) Use induction on ε . Let $e_1 \in E$. If $G \cdot e_1$ is a critical block, then $G \cdot e_1$ has a vertex of degree two and, hence, so does G . If $G \cdot e_1$ is not critical, there is an $e_2 \in E \setminus \{e_1\}$ such that $(G \cdot e_1) - e_2$ is a block. Using the fact that $(G \cdot e_1) - e_2 = (G - e_2) \cdot e_1$, show that e_1 and e_2 are incident with a vertex of degree two in G .

(b) Use (a) and induction on ν .

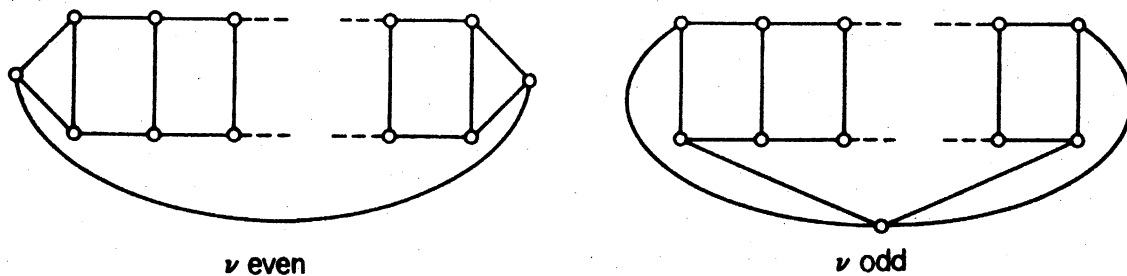
4.1.6 Necessity: if $G - v$ contains a cycle C , consider an Euler tour (with origin v) of the component of $G - E(C)$ that contains v . Sufficiency: let Q be a (v, w) -trail of G which is not an Euler tour. Show that $G - E(Q)$ has exactly one nontrivial component.

4.2.4 Form a new graph G' by adding a new vertex and joining it to every vertex of G . Show that G has a Hamilton path if and only if G' has a Hamilton cycle, and apply theorem 4.5.

4.2.6 Form a new graph G' by adding edges so that $G'[X]$ is complete. Show that G is hamiltonian if and only if G' is hamiltonian, and apply theorem 4.5.

4.2.9 Let P be a longest path in G . If P has length $l < 2\delta$, show, using the proof technique of theorem 4.3, that G has a cycle of length $l + 1$. Now use the fact that G is connected to obtain a contradiction.

4.2.11(b)



ν even

ν odd

- 4.2.13 Use the fact that the Petersen graph is hypohamiltonian (exercise 4.2.12).
- 4.4.1 Consider an Euler tour Q in the weighted graph formed from T by duplicating each of its edges. Now make use of triangle inequalities to obtain from Q a Hamilton cycle in G of weight at most $w(Q)$.
- 5.1.5(a) To show that K_{2n} is 1-factorable, arrange the vertices in the form of a regular $(2n-1)$ -gon with one vertex in the centre. A radial edge together with the edges perpendicular to it is a perfect matching.
- 5.1.6 Label the vertices $0, 1, 2, \dots, 2n$ and arrange the vertices $1, 2, \dots, 2n$ in a circle with 0 at the centre. Let $C = (0, 1, 2, 2n, 3, 2n-1, 4, 2n-2, \dots, n+2, n+1, 0)$ and consider the rotations of C .
- 5.2.3(b) Let G be a $2k$ -regular graph with $V = \{v_1, v_2, \dots, v_\nu\}$; without loss of generality, assume that G is connected. Let C be an Euler tour in G . Form a bipartite graph G' with bipartition (X, Y) , where $X = \{x_1, x_2, \dots, x_\nu\}$ and $Y = \{y_1, y_2, \dots, y_\nu\}$ by joining x_i to y_j whenever v_i immediately precedes v_j on C . Show that G' is 1-factorable and hence that G is 2-factorable.
- 5.2.8 Construct a bipartite graph G with bipartition (X, Y) in which X is the set of rows of \mathbf{Q} , Y is the set of columns of \mathbf{Q} , and row i is joined to column j if and only if the entry q_{ij} is positive. Show that G has a perfect matching, and then use induction on the number of nonzero entries of \mathbf{Q} .
- 5.3.1 Let G be a bipartite graph with bipartition (X, Y) . Assume that ν is even (the case when ν is odd requires a little modification). Obtain a graph H from G by joining all pairs of vertices in Y . G has a matching that saturates every vertex in X if and only if H has a perfect matching.
- 5.3.4 Let G^* be a maximal spanning supergraph of G such that the number of edges in a maximum matching of G^* is the same as for G . Show, using the proof technique of theorem 5.4, that if U is the set of vertices of degree $\nu-1$ in G^* then $G^* - U$ is a disjoint union of complete graphs.
- 6.2.1 See the hint to exercise 5.1.5a.
- 6.2.8 Use the proof technique of theorem 6.2.
- 7.1.3(b) Let $v_1 v_2 \dots v_n$ be a longest path in G . Show that $G - v_2$ has at most one nontrivial component, and use induction on ε .
- 7.2.6(b) Let $p(m-1) = n-1$. The complete $(p+1)$ -partite graph with $m-1$ vertices in each part shows that $r(T, K_{1,n}) > (p+1)(m-1) = m+n-2$. To prove that $r(T, K_{1,n}) \leq m+n-1$, show that any simple graph G with $\delta \geq m-1$ contains every tree T on m vertices.

- (c) The complete $(n-1)$ -partite graph with $m-1$ vertices in each part shows that $r(T, K_n) > (m-1)(n-1)$. To prove that $r(T, K_n) \leq (m-1)(n-1) + 1$, use induction on n and the fact that any simple graph with $\delta \geq m-1$ contains every tree T on m vertices.
- 7.3.3(c) Assume G contains no triangle. Choose a shortest odd cycle C in G . Show that each vertex in $V(G) \setminus V(C)$ can be joined to at most two vertices of C . Apply exercise 7.3.3a to $G - V(C)$, and obtain a contradiction.
- 7.3.4(a) G contains $K_{2,m}$ if and only if there are m vertices with a pair of common neighbours. Any vertex v has $\binom{d(v)}{2}$ pairs of neighbours. Therefore if $\sum_{v \in V} \binom{d(v)}{2} > (m-1) \binom{v}{2}$, G contains $K_{2,m}$.
- 7.5.1 Define a graph G by $V(G) = \{x_1, \dots, x_n\}$, and $E(G) = \{x_i x_j \mid d(x_i, x_j) = 1\}$, and show that if all edges of G are drawn as straight line segments, then (i) any two edges of G are either adjacent or cross, and (ii) if some vertex of G has degree greater than two, it is adjacent to a vertex of degree one. Then prove (a) by induction on n .
- 8.1.6 Let $\mathcal{C} = (V_1, V_2, \dots, V_k)$ be a k -colouring of G , and let \mathcal{C}' be a colouring of G in which each colour class contains at least two vertices. If $|V_i| \geq 2$ for all i , there is nothing to prove, so assume that $V_1 = \{v_1\}$. Let $u_2 \in V_2$ be a vertex of the same colour as v_1 in \mathcal{C}' . Clearly $|V_2| \geq 2$. If $|V_2| > 2$, transfer u_2 to V_1 . Otherwise, let v_2 be the other vertex in V_2 . In \mathcal{C}' , v_1 and v_2 must be assigned different colours. Let $u_3 \in V_3$ be a vertex of the same colour as v_2 in \mathcal{C}' . As before, $|V_3| \geq 2$. Proceeding in this way, one must eventually find a set V_i with $|V_i| > 2$. G can now be recoloured so that fewer colour classes contain only one vertex.
- 8.1.13(a) Let (X_1, X_2, \dots, X_n) and (Y_1, Y_2, \dots, Y_n) be n -colourings of $G[X]$ and $G[Y]$, respectively. Construct a bipartite graph H with bipartition $(\{x_1, x_2, \dots, x_n\}, \{y_1, y_2, \dots, y_n\})$ by joining x_i and y_j if and only if the edge cut $[X_i, Y_j]$ is empty in G . Using exercise 5.2.6b, show that H has a perfect matching. If x_i is matched with $y_{f(i)}$ under this matching, let $V_i = X_i \cup Y_{f(i)}$. Show that (V_1, V_2, \dots, V_n) is an n -colouring of G .
- 8.3.1 Show that it suffices to consider 2-connected graphs. Choose a longest cycle C in G and show that there are two paths across C as in theorem 8.5.
- 8.3.2(a) If $\delta \geq 3$, use exercise 8.3.1. If there is a vertex of degree less than three, delete it and use induction.

- 8.4.8 Consider the expansion of $\pi_k(G)$ in terms of chromatic polynomials of complete graphs.
- 8.5.2(a) It is easily verified that H has girth at least six. If H is k -colourable, there is a ν -element subset of S all of whose members receive the same colour. Consider the corresponding copy of G and obtain a contradiction.
- 9.2.8 The dual G^* is 2-edge-connected and 3-regular and, hence (corollary 5.4), has a perfect matching M . $(G^* \cdot M)^*$ is a bipartite subgraph of G .
- 10.2.2 Form a new digraph on the same vertex set joining u to v if v is reachable from u , and apply corollary 10.1.
- 10.2.5 Let D_1 and D_2 be the spanning subdigraphs of D such that the arcs of D_1 are the arcs (u, v) of D for which $f(u) \leq f(v)$, and the arcs of D_2 are the arcs (u, v) for which $f(u) > f(v)$. Show that either $\chi(D_1) > m$ or $\chi(D_2) > n$, and apply theorem 10.1.
- 10.3.4 Let $v_1 v_2 \dots v_{2n+1} v_1$ be an odd cycle. If $(v_i, v_{i+1}) \in A$, set $P_i = (v_i, v_{i+1})$; if $(v_i, v_{i+1}) \notin A$, let P_i be a directed (v_i, v_{i+1}) -path. If some P_i is of even length, $P_i + (v_{i+1}, v_i)$ is a directed odd cycle; otherwise, $P_1 P_2 \dots P_{2n+1}$ is a closed directed trail of odd length, and therefore contains a directed odd cycle.
- 11.3.5 Use the construction given in the proof of theorem 11.6, and assign capacity $m(v)$ to arc (v', v'') .
- 11.4.4 Use induction on k and exercise 11.4.3.
- 11.5.4 Use an argument similar to that in exercise 1.5.7.
- 11.5.5(a) Necessity follows on taking V_1 as the set of vertices with indegree m and V_2 as the set of vertices with indegree n . To prove sufficiency, construct a network N by forming the associated digraph of G , assigning unit capacity to each arc, and regarding the elements of V_1 as sources and the elements of V_2 as sinks. By theorem 11.8, there is a flow f in N (which can be assumed integral) in which the supply at each source and demand at each sink is $|m - n|$. The f -saturated arcs induce an (m, n) -orientation on a subgraph H of G . An (m, n) -orientation of G can now be obtained by giving the remaining edges an eulerian orientation.
- 12.2.1(a) Use induction on the order of the submatrix. Let \mathbf{P} be a square submatrix. If each column of \mathbf{P} contains two nonzero entries, then $\det \mathbf{P} = 0$. Otherwise, expand $\det \mathbf{P}$ about a column with exactly one nonzero entry, and apply the induction hypothesis.
- 12.3.3 Show, first, that in any perfect rectangle the smallest constituent square is not on the boundary of the rectangle. Now suppose that there is a perfect cube and consider the perfect square induced on the base of this cube by the constituent cubes.