Appendix I
Hints to Starred Exercises

1.2.9(*b*) If $G \not\cong T_{m,n}$, then G has parts of size $n_1, n_2, \ldots, n_m$, with $n_i - n_j > 1$ for some i and j. Show that the complete m-partite graph with parts of size $n_1, n_2, \ldots, n_i - 1, \ldots, n_j + 1, \ldots, n_m$ has more edges than G.

1.3.3 In terms of the adjacency matrix $\mathbf{A}$, an automorphism of G is a permutation matrix $\mathbf{P}$ such that $\mathbf{PAP}' = \mathbf{A}$ or, equivalently, $\mathbf{PA} = \mathbf{AP}$ (since $\mathbf{P}' = \mathbf{P}^{-1}$). Show that if $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ belonging to an eigenvalue λ, then, for any automorphism $\mathbf{P}$ of G, so is $\mathbf{Px}$. Since the eigenvalues of $\mathbf{A}$ are distinct and $\mathbf{P}$ is orthogonal, $\mathbf{P}^2\mathbf{x} = \mathbf{x}$ for all eigenvectors $\mathbf{x}$.

1.4.5 Suppose that all induced subgraphs of G on n vertices have m edges. Show that, for any two vertices v_i and v_j,

$$\varepsilon(G) - d(v_i) = \varepsilon(G - v_i) = m\binom{\nu-1}{n}\Big/\binom{\nu-3}{n-2}$$

$$\varepsilon(G) - d(v_i) - d(v_j) + a_{ij} = \varepsilon(G - v_i - v_j) = m\binom{\nu-2}{n}\Big/\binom{\nu-4}{n-2}$$

where $a_{ij} = 1$ or 0 according as v_i and v_j are adjacent or not. Deduce that a_{ij} is independent of i and j.

1.5.7(*a*) To prove the necessity, first show that if G is simple with $u_1v_1, u_2v_2 \in E$ and $u_1v_2, u_2v_1 \notin E$, then $G - \{u_1v_1, u_2v_2\} + \{u_1v_2, u_2v_1\}$ has the same degree sequence as G. Using this, show that if $\mathbf{d}$ is graphic, then there is a simple graph G with $V = \{v_1, v_2, \ldots, v_n\}$ such that (i) $d(v_i) = d_i$ for $1 \le i \le n$, and (ii) v_1 is joined to $v_2, v_3, \ldots, v_{d_1+1}$. The graph $G - v_1$ has degree sequence $\mathbf{d}'$.

1.5.8 Show that a bipartite subgraph with the largest possible number of edges has this property.

1.5.9 Define a graph on S in which x_i and x_j are adjacent if and only if they are at distance one. Show that in this graph each vertex has degree at most six.

1.7.3 Consider a longest path and the vertices adjacent to the origin of this path.

1.7.6(*b*) By contradiction. Let G be a smallest counter-example. Show that (i) the girth of G is at least five, and (ii) $\delta \ge 3$. Deduce that $\nu \le 8$ and show that no such graph exists.

2.1.10 To prove the sufficiency, consider a graph G with degree sequence $\mathbf{d} = (d_1, d_2, \ldots, d_\nu)$ and as few components as possible. If

G is not connected, show that, by a suitable exchange of edges (as in the hint to exercise 1.5.7a), there is a graph with degree sequence $\mathbf{d}$ and fewer components than G.

2.2.12 Define a labelled graph G as follows: the vertices of G are the subsets $A_1, A_2, \ldots, A_n$, and A_i is joined to A_j $(i \neq j)$ by an edge labelled a if either $A_i = A_j \cup \{a\}$ or $A_j = A_i \cup \{a\}$. For any subgraph H of G, let $L(H)$ be the set of labels on edges of H. Show that if F is a maximal forest of G, then $L(F) = L(G)$. Any element x in $S \backslash L(F)$ has the required property.

2.4.2 Several applications of theorem 2.8 yield the recurrence relation

$$w_n - 4w_{n-1} + 4w_{n-2} - 1 = 0$$

where w_n is the number of spanning trees in the wheel with n spokes. Solve this recurrence relation.

3.2.6 Form a new graph G' by adding two vertices x and y, and joining x to all vertices in X and y to all vertices in Y. Show that G' is 2-connected and apply theorem 3.2.

3.2.7(a) Use induction on ε. Let $e_1 \in E$. If $G \cdot e_1$ is a critical block, then $G \cdot e_1$ has a vertex of degree two and, hence, so does G. If $G \cdot e_1$ is not critical, there is an $e_2 \in E \backslash \{e_1\}$ such that $(G \cdot e_1) - e_2$ is a block. Using the fact that $(G \cdot e_1) - e_2 = (G - e_2) \cdot e_1$, show that e_1 and e_2 are incident with a vertex of degree two in G.

(b) Use (a) and induction on ν.

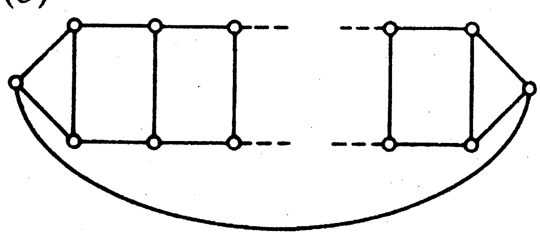
4.1.6 Necessity: if $G - v$ contains a cycle C, consider an Euler tour (with origin v) of the component of $G - E(C)$ that contains v. Sufficiency: let Q be a (v, w)-trail of G which is not an Euler tour. Show that $G - E(Q)$ has exactly one nontrivial component.

4.2.4 Form a new graph G' by adding a new vertex and joining it to every vertex of G. Show that G has a Hamilton path if and only if G' has a Hamilton cycle, and apply theorem 4.5.

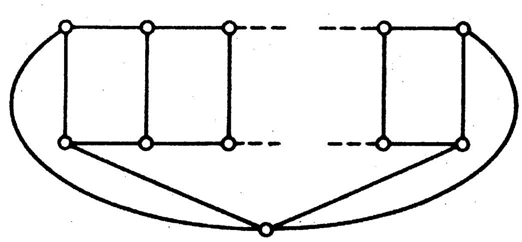
4.2.6 Form a new graph G' by adding edges so that $G'[X]$ is complete. Show that G is hamiltonian if and only if G' is hamiltonian, and apply theorem 4.5.

4.2.9 Let P be a longest path in G. If P has length $l < 2\delta$, show, using the proof technique of theorem 4.3, that G has a cycle of length $l+1$. Now use the fact that G is connected to obtain a contradiction.

4.2.11(b)



ν even



ν odd

4.2.13 Use the fact that the Petersen graph is hypohamiltonian (exercise 4.2.12).

4.4.1 Consider an Euler tour Q in the weighted graph formed from T by duplicating each of its edges. Now make use of triangle inequalities to obtain from Q a Hamilton cycle in G of weight at most $w(Q)$.

5.1.5(a) To show that K_{2n} is 1-factorable, arrange the vertices in the form of a regular $(2n-1)$-gon with one vertex in the centre. A radial edge together with the edges perpendicular to it is a perfect matching.

5.1.6 Label the vertices $0, 1, 2, \ldots, 2n$ and arrange the vertices 1, $2, \ldots, 2n$ in a circle with 0 at the centre. Let $C = (0, 1, 2, 2n, 3, 2n-1, 4, 2n-2, \ldots, n+2, n+1, 0)$ and consider the rotations of C.

5.2.3(b) Let G be a $2k$-regular graph with $V = \{v_1, v_2, \ldots, v_\nu\}$; without loss of generality, assume that G is connected. Let C be an Euler tour in G. Form a bipartite graph G' with bipartition (X, Y), where $X = \{x_1, x_2, \ldots, x_\nu\}$ and $Y = \{y_1, y_2, \ldots, y_\nu\}$ by joining x_i to y_j whenever v_i immediately precedes v_j on C. Show that G' is 1-factorable and hence that G is 2-factorable.

5.2.8 Construct a bipartite graph G with bipartition (X, Y) in which X is the set of rows of $\mathbf{Q}$, Y is the set of columns of $\mathbf{Q}$, and row i is joined to column j if and only if the entry q_{ij} is positive. Show that G has a perfect matching, and then use induction on the number of nonzero entries of $\mathbf{Q}$.

5.3.1 Let G be a bipartite graph with bipartition (X, Y). Assume that ν is even (the case when ν is odd requires a little modification). Obtain a graph H from G by joining all pairs of vertices in Y. G has a matching that saturates every vertex in X if and only if H has a perfect matching.

5.3.4 Let G^* be a maximal spanning supergraph of G such that the number of edges in a maximum matching of G^* is the same as for G. Show, using the proof technique ot theorem 5.4, that if U is the set of vertices of degree $\nu - 1$ in G^* then $G^* - U$ is a disjoint union of complete graphs.

6.2.1 See the hint to exercise 5.1.5a.

6.2.8 Use the proof technique of theorem 6.2.

7.1.3(b) Let $v_1v_2 \ldots v_n$ be a longest path in G. Show that $G - v_2$ has at most one nontrivial component, and use induction on ε.

7.2.6(b) Let $p(m-1) = n-1$. The complete $(p+1)$-partite graph with $m-1$ vertices in each part shows that $r(T, K_{1,n}) > (p+1)(m-1) = m+n-2$. To prove that $r(T, K_{1,n}) \le m+n-1$, show that any simple graph G with $\delta \ge m-1$ contains every tree T on m vertices.

(c) The complete $(n-1)$-partite graph with $m-1$ vertices in each part shows that $r(T, K_n) > (m-1)(n-1)$. To prove that $r(T, K_n) \leq (m-1)(n-1)+1$, use induction on n and the fact that any simple graph with $\delta \geq m-1$ contains every tree T on m vertices.

7.3.3(c) Assume G contains no triangle. Choose a shortest odd cycle C in G. Show that each vertex in $V(G)\backslash V(C)$ can be joined to at most two vertices of C. Apply exercise 7.3.3a to $G - V(C)$, and obtain a contradiction.

7.3.4(a) G contains $K_{2,m}$ if and only if there are m vertices with a pair of common neighbours. Any vertex v has $\binom{d(v)}{2}$ pairs of neighbours. Therefore if $\sum_{v \in V} \binom{d(v)}{2} > (m-1)\binom{\nu}{2}$, G contains $K_{2,m}$.

7.5.1 Define a graph G by $V(G) = \{x_1, \ldots, x_n\}$, and $E(G) = \{x_i x_j \mid d(x_i, x_j) = 1\}$, and show that if all edges of G are drawn as straight line segments, then (i) any two edges of G are either adjacent or cross, and (ii) if some vertex of G has degree greater than two, it is adjacent to a vertex of degree one. Then prove (a) by induction on n.

8.1.6 Let $\mathscr{C} = (V_1, V_2, \ldots, V_k)$ be a k-colouring of G, and let $\mathscr{C}'$ be a colouring of G in which each colour class contains at least two vertices. If $|V_i| \geq 2$ for all i, there is nothing to prove, so assume that $V_1 = \{v_1\}$. Let $u_2 \in V_2$ be a vertex of the same colour as v_1 in $\mathscr{C}'$. Clearly $|V_2| \geq 2$. If $|V_2| > 2$, transfer u_2 to V_1. Otherwise, let v_2 be the other vertex in V_2. In $\mathscr{C}'$, v_1 and v_2 must be assigned different colours. Let $u_3 \in V_3$ be a vertex of the same colour as v_2 in $\mathscr{C}'$. As before, $|V_3| \geq 2$. Proceeding in this way, one must eventually find a set V_i with $|V_i| > 2$. G can now be recoloured so that fewer colour classes contain only one vertex.

8.1.13(a) Let $(X_1, X_2, \ldots, X_n)$ and $(Y_1, Y_2, \ldots, Y_n)$ be n-colourings of $G[X]$ and $G[Y]$, respectively. Construct a bipartite graph H with bipartition $(\{x_1, x_2, \ldots, x_n\}, \{y_1, y_2, \ldots, y_n\})$ by joining x_i and y_j if and only if the edge cut $[X_i, Y_j]$ is empty in G. Using exercise 5.2.6b, show that H has a perfect matching. If x_i is matched with $y_{f(i)}$ under this matching, let $V_i = X_i \cup Y_{f(i)}$. Show that $(V_1, V_2, \ldots, V_n)$ is an n-colouring of G.

8.3.1 Show that it suffices to consider 2-connected graphs. Choose a longest cycle C in G and show that there are two paths across C as in theorem 8.5.

8.3.2(a) If $\delta \geq 3$, use exercise 8.3.1. If there is a vertex of degree less than three, delete it and use induction.

8.4.8 Consider the expansion of $\pi_k(G)$ in terms of chromatic polynomials of complete graphs.

8.5.2(a) It is easily verified that H has girth at least six. If H is k-colourable, there is a ν-element subset of S all of whose members receive the same colour. Consider the corresponding copy of G and obtain a contradiction.

9.2.8 The dual G^* is 2-edge-connected and 3-regular and, hence (corollary 5.4), has a perfect matching M. $(G^* \cdot M)^*$ is a bipartite subgraph of G.

10.2.2 Form a new digraph on the same vertex set joining u to v if v is reachable from u, and apply corollary 10.1.

10.2.5 Let D_1 and D_2 be the spanning subdigraphs of D such that the arcs of D_1 are the arcs (u, v) of D for which $f(u) \leq f(v)$, and the arcs of D_2 are the arcs (u, v) for which $f(u) > f(v)$. Show that either $\chi(D_1) > m$ or $\chi(D_2) > n$, and apply theorem 10.1.

10.3.4 Let $v_1 v_2 \ldots v_{2n+1} v_1$ be an odd cycle. If $(v_i, v_{i+1}) \in A$, set $P_i = (v_i, v_{i+1})$; if $(v_i, v_{i+1}) \notin A$, let P_i be a directed (v_i, v_{i+1})-path. If some P_i is of even length, $P_i + (v_{i+1}, v_i)$ is a directed odd cycle; otherwise, $P_1 P_2 \ldots P_{2n+1}$ is a closed directed trail of odd length, and therefore contains a directed odd cycle.

11.3.5 Use the construction given in the proof of theorem 11.6, and assign capacity $m(v)$ to arc (v', v'').

11.4.4 Use induction on k and exercise 11.4.3.

11.5.4 Use an argument similar to that in exercise 1.5.7.

11.5.5(a) Necessity follows on taking V_1 as the set of vertices with indegree m and V_2 as the set of vertices with indegree n. To prove sufficiency, construct a network N by forming the associated digraph of G, assigning unit capacity to each arc, and regarding the elements of V_1 as sources and the elements of V_2 as sinks. By theorem 11.8, there is a flow f in N (which can be assumed integral) in which the supply at each source and demand at each sink is $|m-n|$. The f-saturated arcs induce an (m,n)-orientation on a subgraph H of G. An (m,n)-orientation of G can now be obtained by giving the remaining edges an eulerian orientation.

12.2.1(a) Use induction on the order of the submatrix. Let $\mathbf{P}$ be a square submatrix. If each column of $\mathbf{P}$ contains two nonzero entries, then $\det \mathbf{P} = 0$. Otherwise, expand $\det \mathbf{P}$ about a column with exactly one nonzero entry, and apply the induction hypothesis.

12.3.3 Show, first, that in any perfect rectangle the smallest constituent square is not on the boundary of the rectangle. Now suppose that there is a perfect cube and consider the perfect square induced on the base of this cube by the constituent cubes.