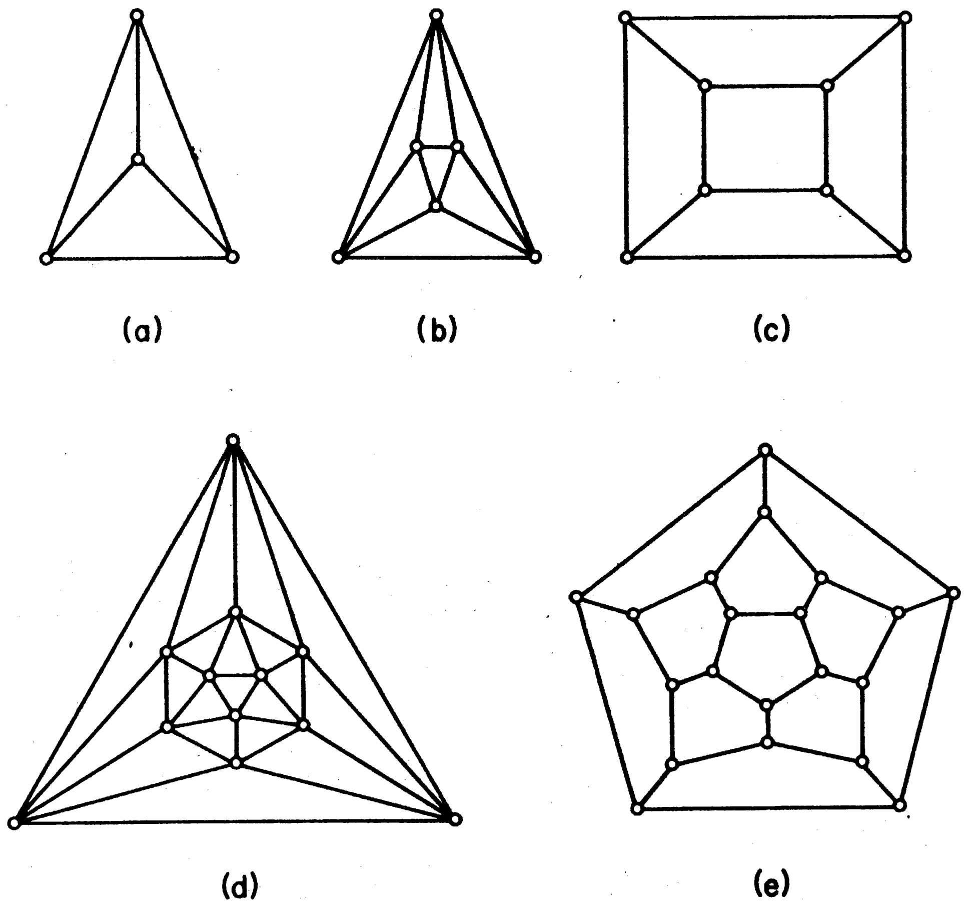
Appendix III
Some Interesting Graphs

There are a number of graphs which are interesting because of their special structure. We have already met some of these (for example, the Grinberg graph, the Grötzsch graph, the Herschel graph and the Ramsey graphs). Here we present a selection of other interesting graphs and families of graphs.

THE PLATONIC GRAPHS

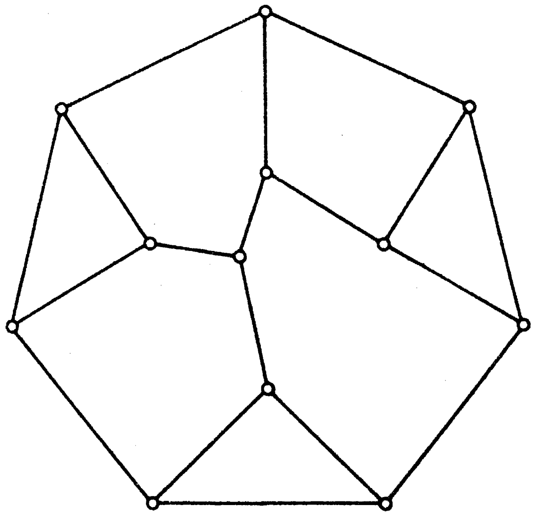
These are the graphs whose vertices and edges are the vertices and edges of the platonic solids (see Fréchet and Fan, 1967).



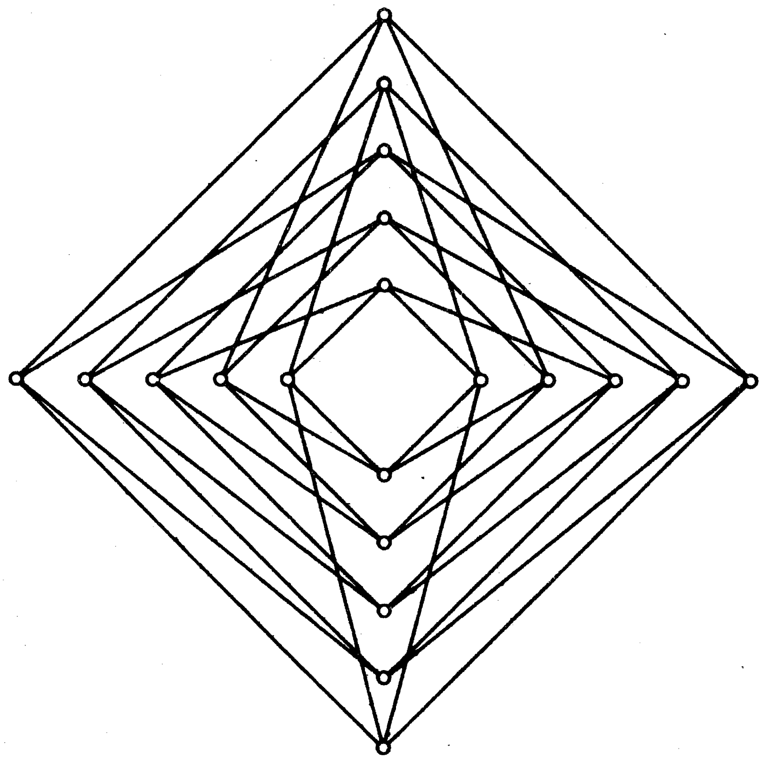
(*a*) The tetrahedron; (*b*) the octahedron; (*c*) the cube; (*d*) the icosahedron; (*e*) the dodecahedron

AUTOMORPHISM GROUPS

(i) As has already been noted (exercise 1.2.12), every group is isomorphic to the automorphism group of some graph. Frucht (1949) showed, in fact, that for any group there is a 3-regular graph with that group. The smallest 3-regular graph whose group is the identity is the following:



(ii) Folkman (1967) proved that every edge- but not vertex-transitive regular graph has at least 20 vertices. This result is best possible:



The Folkman graph

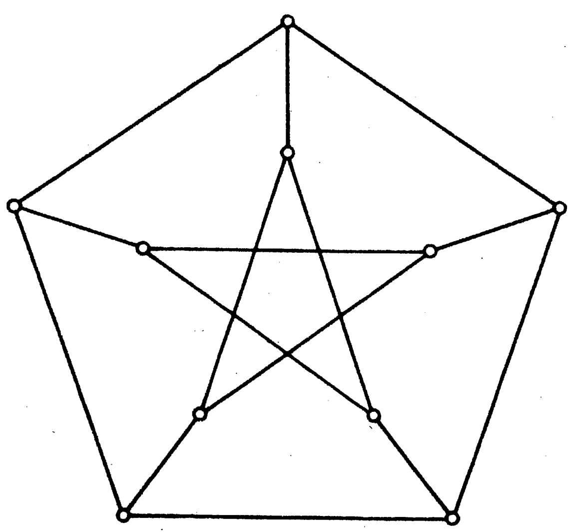
The *Gray graph* (see Bouwer, 1972) is a 3-regular edge- but not vertex-transitive graph on 54 vertices. It has the following description: take three copies of $K_{3,3}$. For a particular edge e, subdivide e in each of the three copies and join the resulting three vertices to a new vertex. Repeat this with each edge.

CAGES

An m-regular graph of girth n with the least possible number of vertices is called an (m, n)-*cage*. If we denote by $f(m, n)$ the number of vertices in an (m, n)-cage, it is easy to see that $f(2, n) = n$ and for $m \geq 3$,

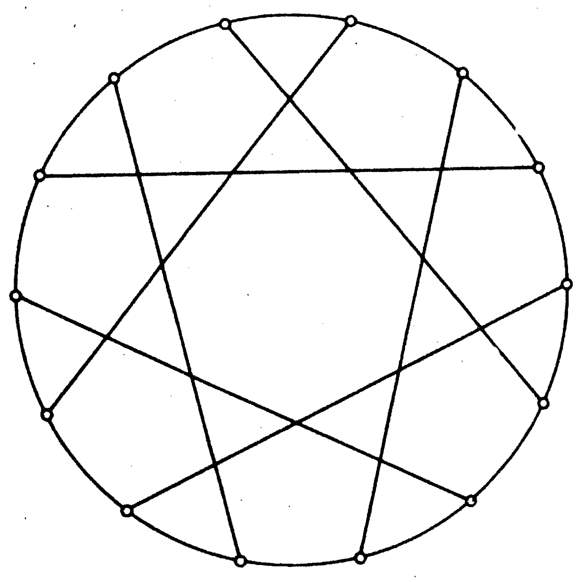
$$f(m, n) \geq \begin{cases} \dfrac{m(m-1)^r - 2}{m-2} & \text{if} \quad n = 2r+1 \\[2ex] \dfrac{2(m-1)^r - 2}{m-2} & \text{if} \quad n = 2r \end{cases} \tag{III.1}$$

The $(2, n)$-cage is the n-cycle, the $(m, 3)$-cage is K_{m+1}, and the $(m, 4)$-cage is $K_{m,m}$. In each of these cases, equality holds in (III.1). It has been shown by Hoffman and Singleton (1960) that, for $m \geq 3$ and $n \geq 5$, equality can hold in (III.1) only if $n = 5$ and $m = 3$, 7 or 57, or $n = 6$, 8 or 12. When $m - 1$ is a prime power, the $(m, 6)$-cage is the point-line incidence graph of the projective plane of order $m - 1$; the $(m, 8)$- and $(m, 12)$-cages are also obtained from projective geometries (see Biggs, 1974 for further details). Some of the smaller (m, n)-cages are depicted below:



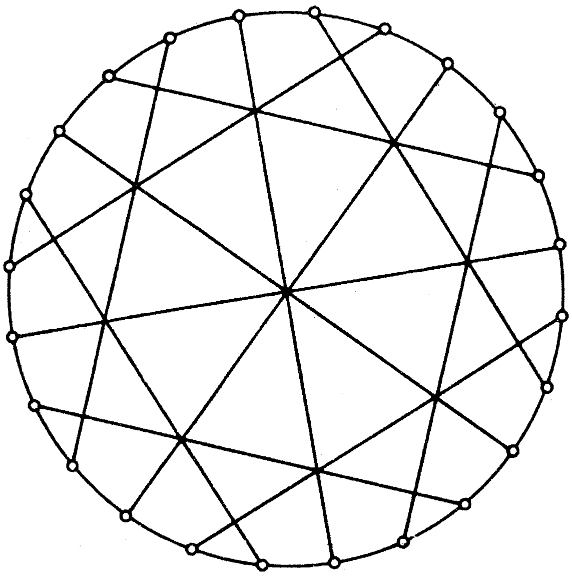
(3,5) - cage

The Petersen graph



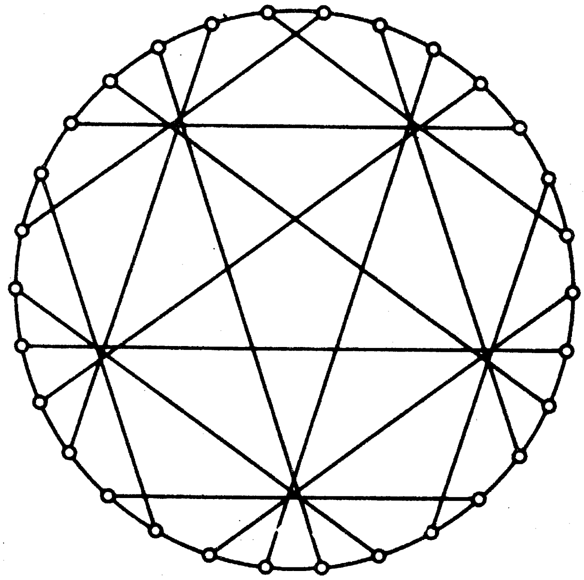
(3,6) - cage

The Heawood graph



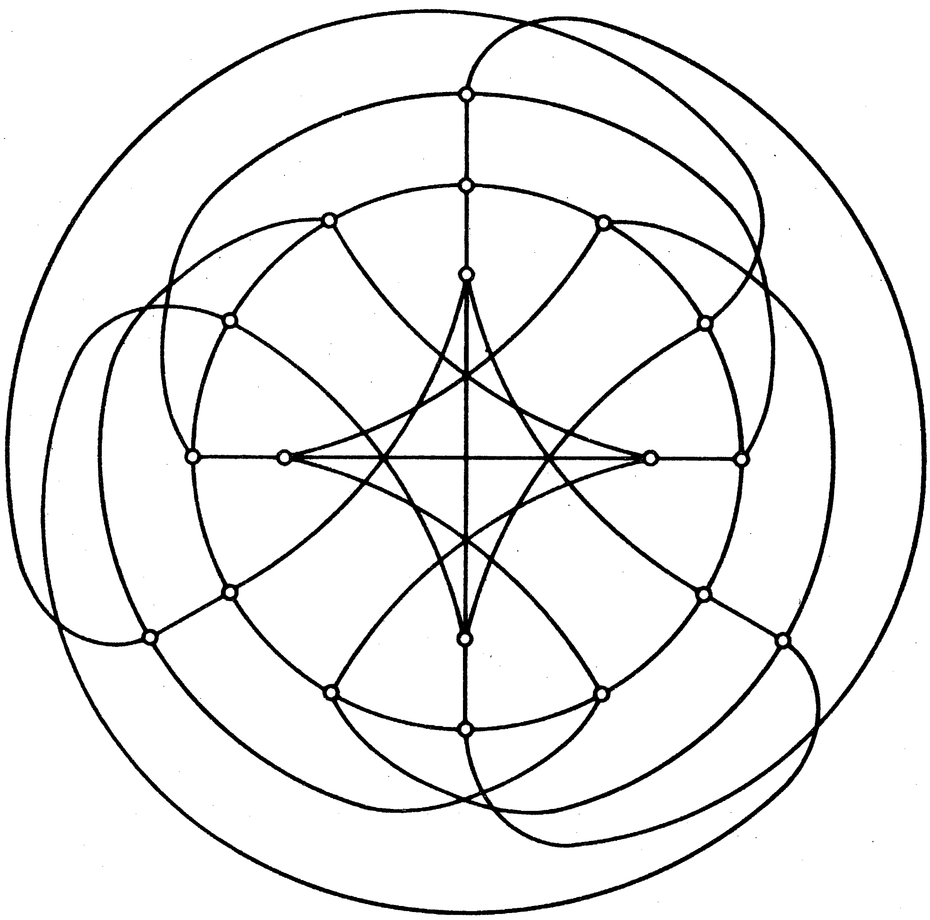
(3,7) - cage

The McGee graph



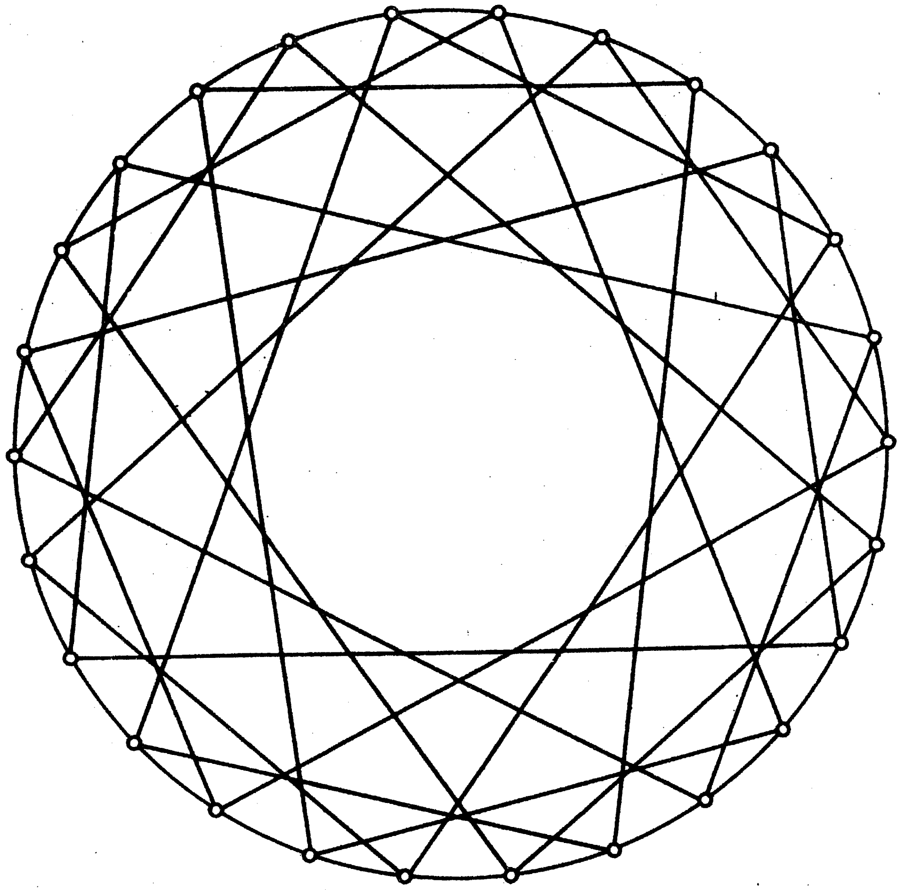
(3,8) - cage

The Tutte–Coxeter graph

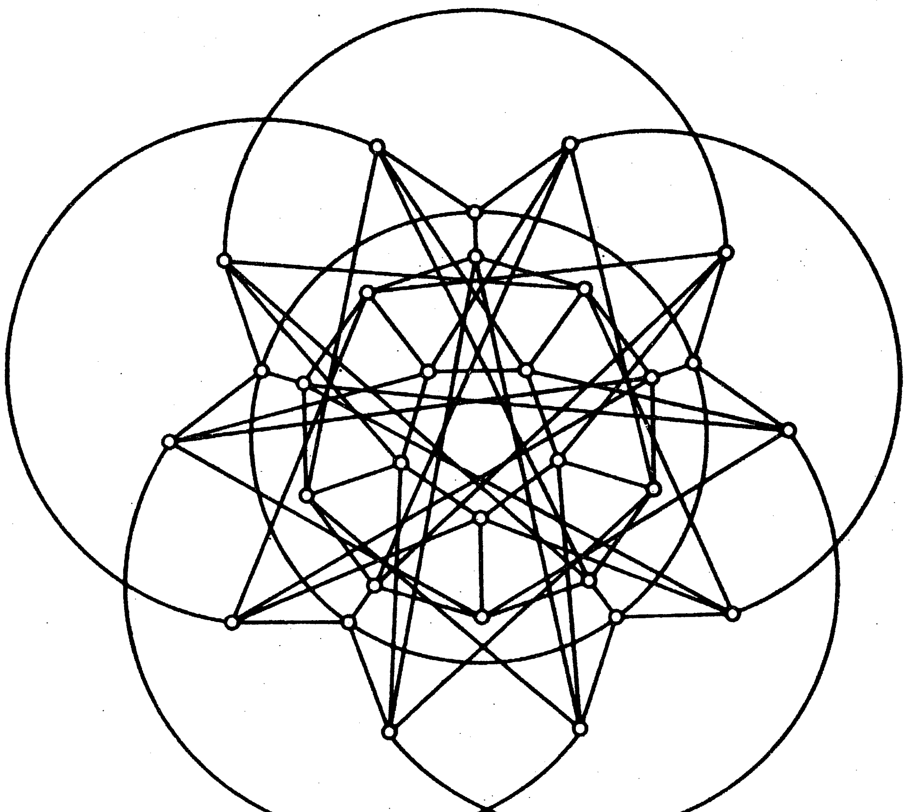


(4,5) - cage

The Robertson graph

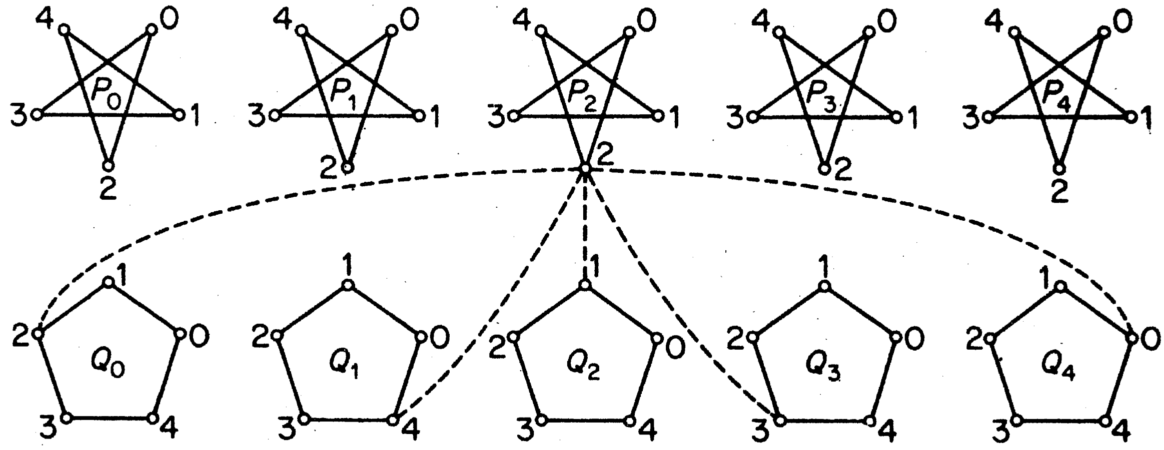


(4,6) – cage



(5,5) – cage
The Robertson–Wegner graph

The $(7,5)$-cage (the *Hoffman–Singleton graph*) can be described as follows: it has ten 5-cycles P_0, P_1, P_2, P_3, P_4, Q_0, Q_1, Q_2, Q_3, Q_4, labelled as shown below; vertex i of P_j is joined to vertex $i+jk$ (mod 5) of Q_k. (For example, vertex 2 of P_2 is connected as indicated.)



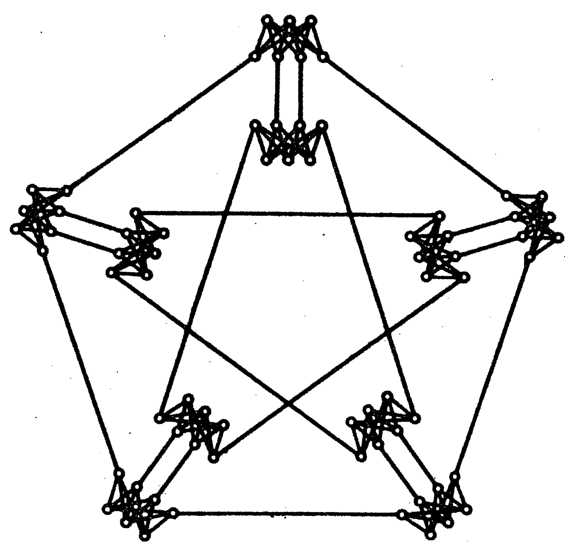
(7,5)–cage

The Hoffman–Singleton graph

NONHAMILTONIAN GRAPHS

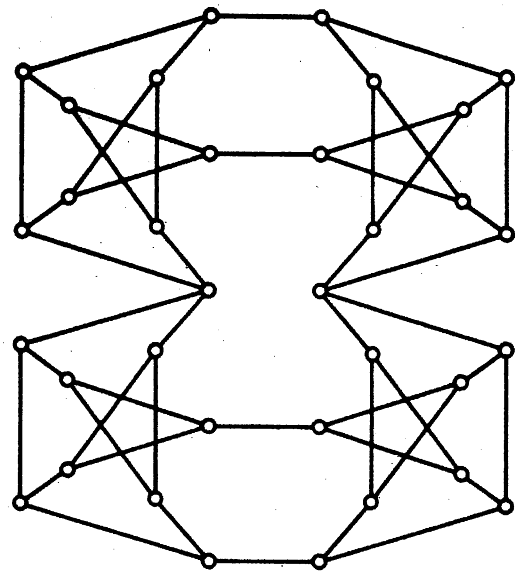
(i) Conditions for a graph to be hamiltonian have been sought ever since Tait made his conjecture on planar graphs. Listed here are counter-examples to several conjectured results.

(a) Every 4-regular 4-connected graph is hamiltonian (C. St. J. A. Nash-Williams).



The Meredith graph

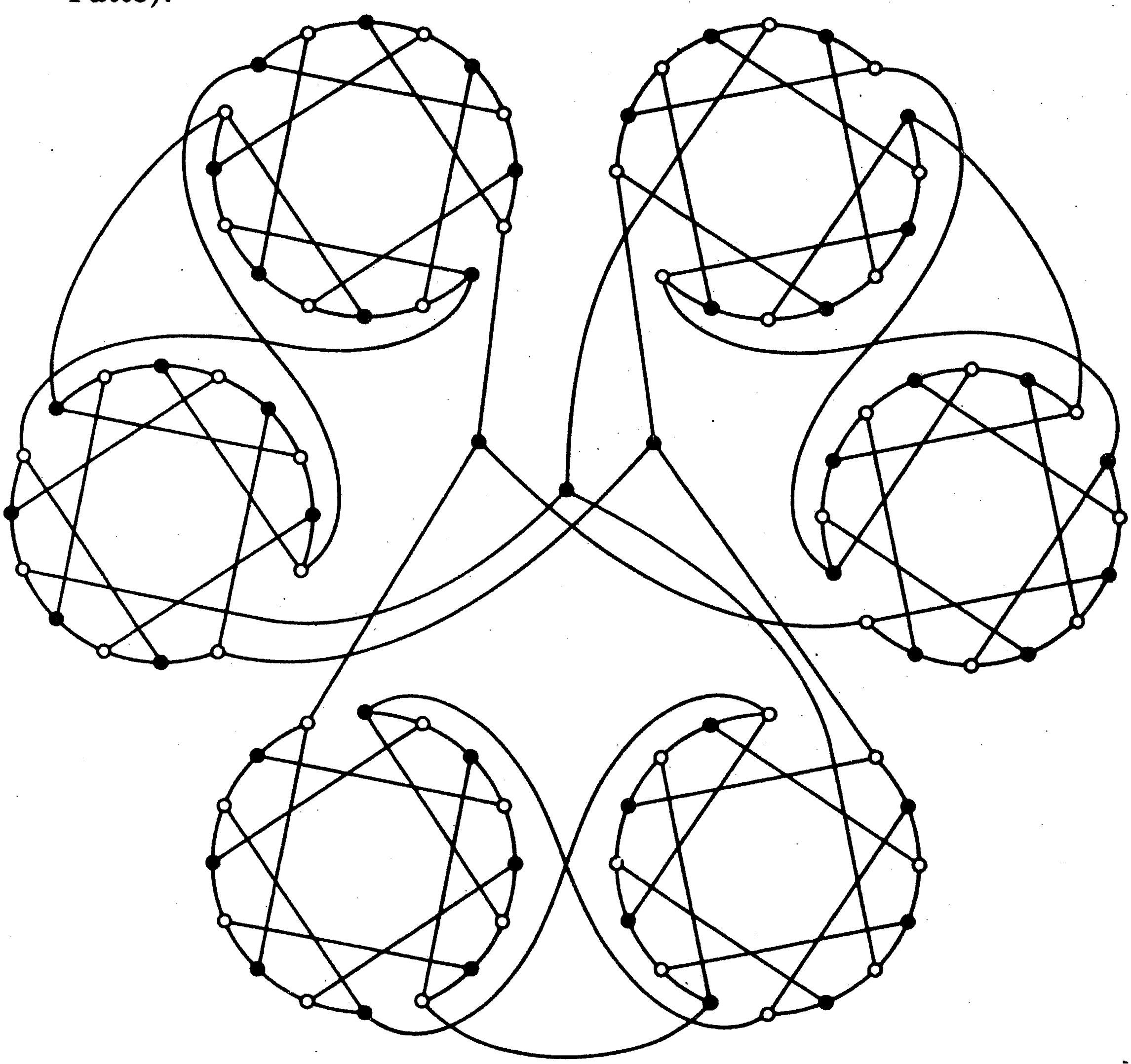
(b) There is no hypotraceable graph (T. Gallai).



The Thomassen graph

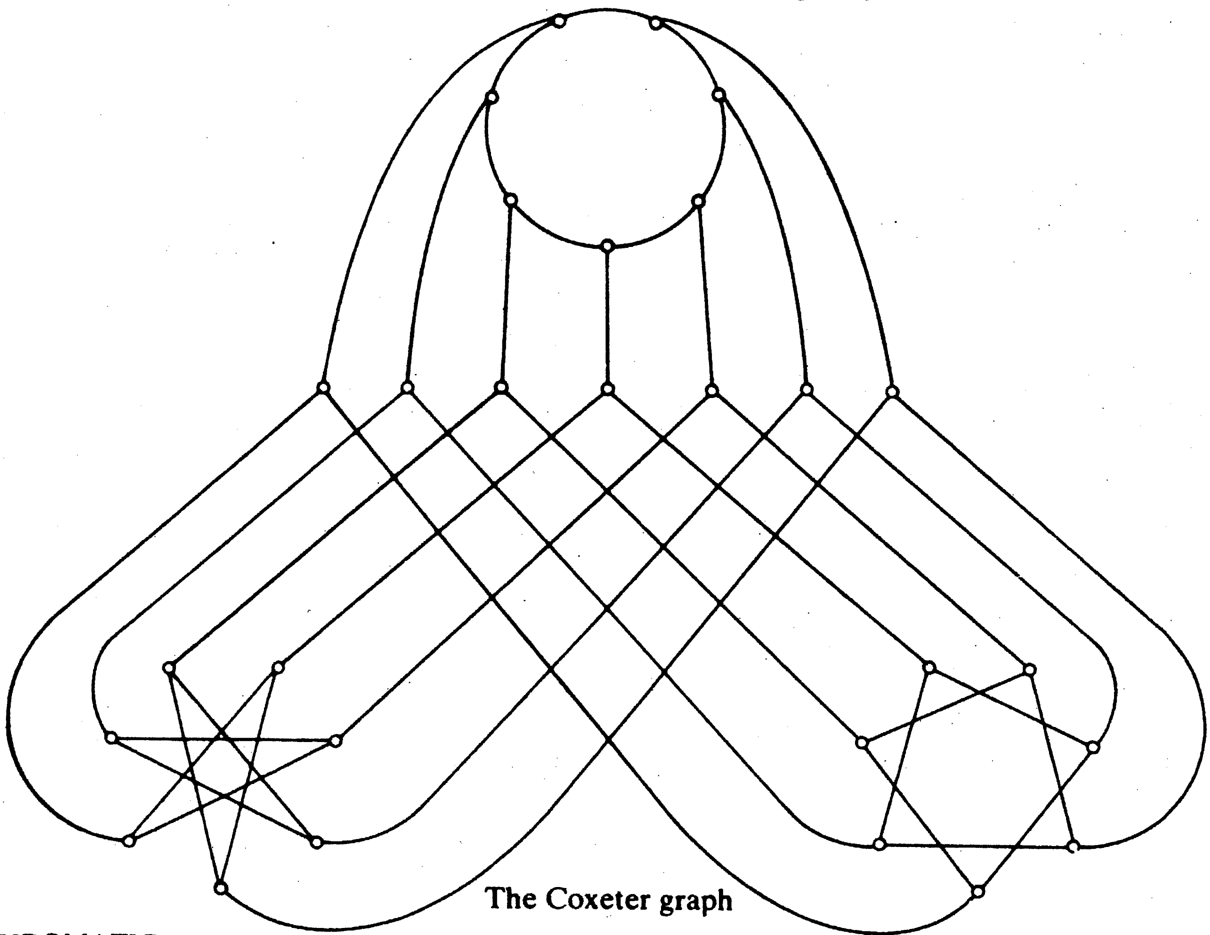
(The first hypotraceable graph was discovered by J. D. Horton.)

(c) Every 3-regular 3-connected bipartite graph is hamiltonian (W. T. Tutte).



The Horton graph

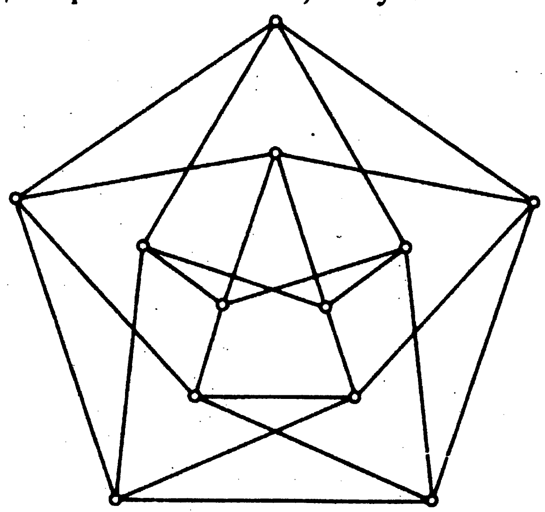
(ii) An example of a nonhamiltonian graph with a high degree of symmetry—there is an automorphism taking any path of length three into any other. (The Petersen graph also has this property.) See Tutte (1960).



The Coxeter graph

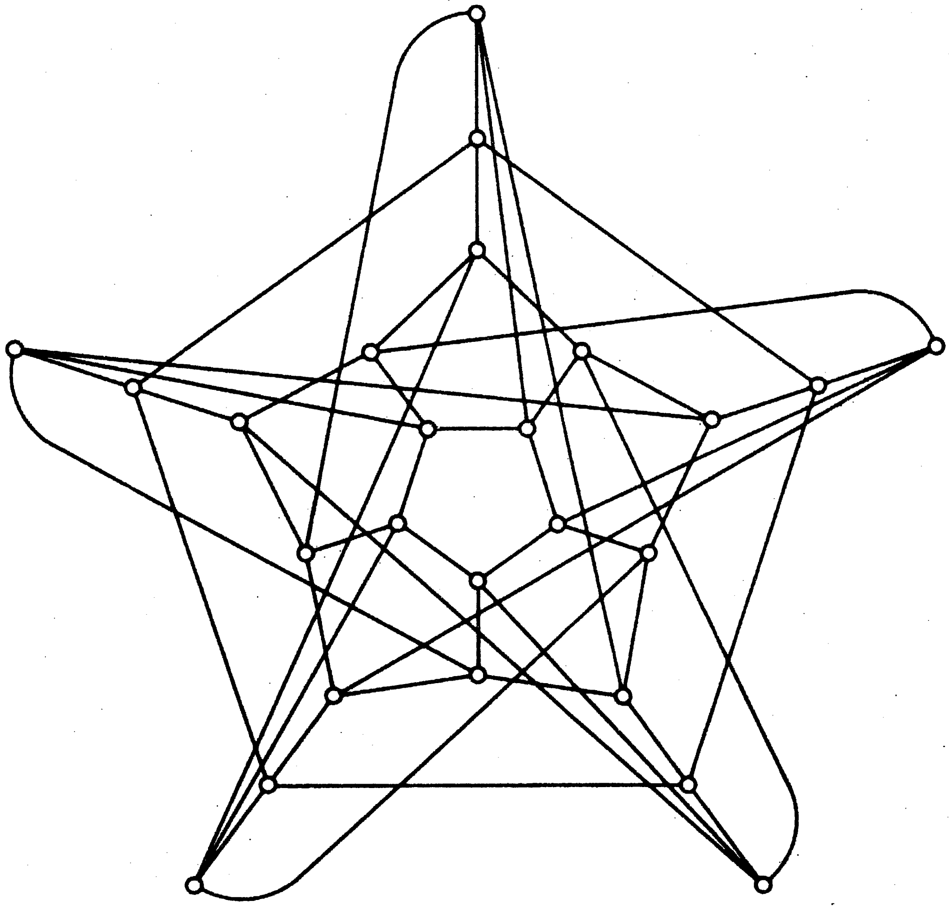
CHROMATIC NUMBER

(i) Grünbaum (1970) has conjectured that, for every $m>1$ and $n>2$, there exists an m-regular, m-chromatic graph of girth at least n. For $n=3$, this is trivial, and for $m=2$ and 3, the validity of the conjecture follows from the existence of the cages†. Apart from this, only two such graphs are known:



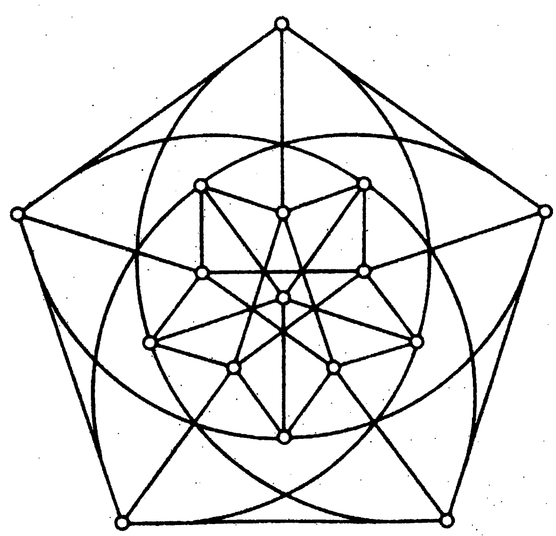
The Chvátal graph

† This conjecture has now been disproved: (Borodin, O. V. and Kostochka, A. V. (1976). On an upper bound of the graph's chromatic number depending on graph's degree and density. *Inst. Maths.*, Novosibirsk, preprint GT-7).



The Grünbaum graph

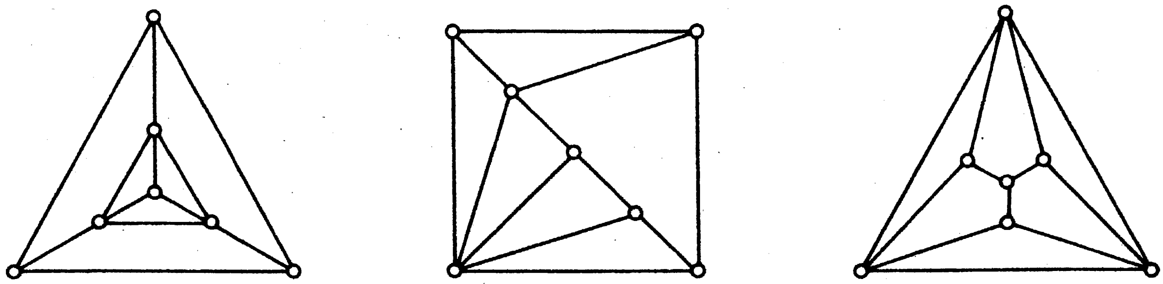
(ii) Since $r(3, 3, 3) = 17$ (see exercise 7.2.3), there is a 3-edge colouring of K_{16} without monochromatic triangles. Kalbfleisch and Stanton (1968) showed that, in such a colouring, the subgraph induced by the edges of any one colour is isomorphic to the following graph:



The Greenwood–Gleason graph

EMBEDDINGS

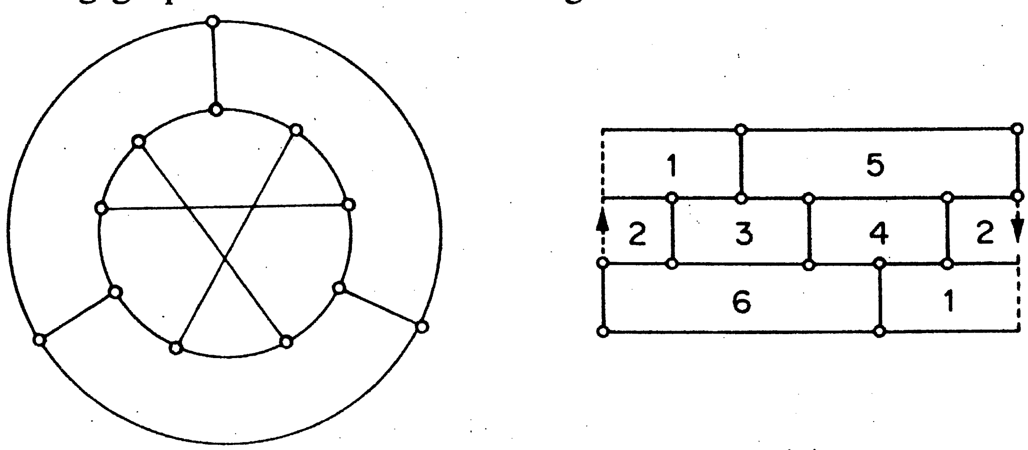
(i) Simple examples of self-dual plane graphs are the wheels. Some more interesting plane graphs with this property are depicted below (see, for example, Smith and Tutte, 1950).



(ii) The *chromatic number* $\chi(S)$ of a surface S is the maximum number of colours required to properly colour the faces of any map on S. (The four-colour conjecture claims that the sphere is 4-chromatic.) Heawood (1890) proved that if S has characteristic $n<2$, then

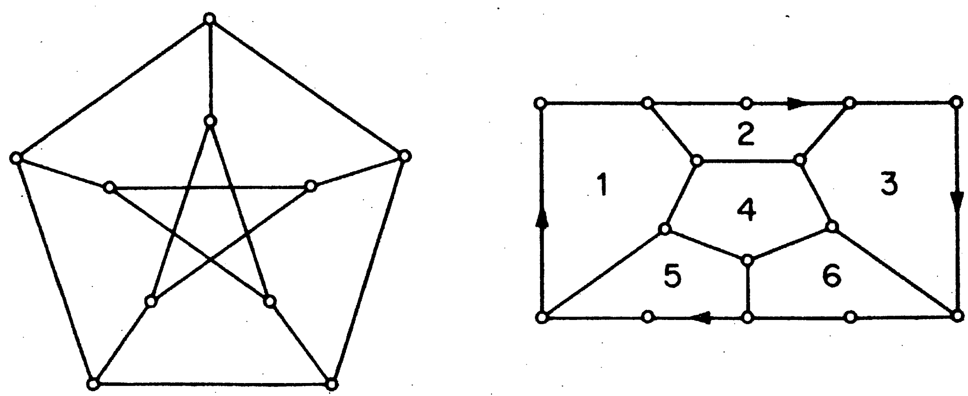
$$\chi(S)\le\left[\tfrac{1}{2}\left(7+\sqrt{49-24n}\right)\right] \tag{III.2}$$

For the projective plane and Möbius band (characteristic 1) and for the torus (characteristic 0), the bound given in (III.2) is attained, as is shown by the following graphs and their embeddings:



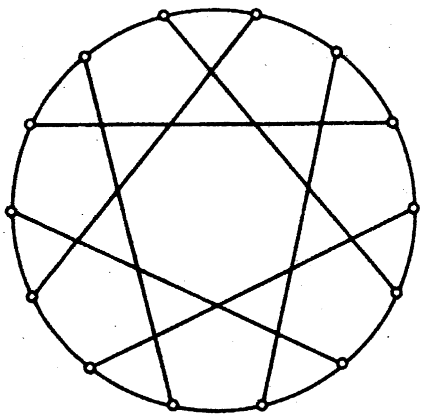
(a) (b)

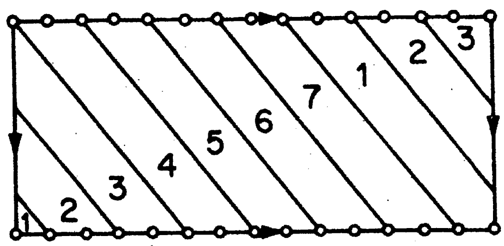
(*a*) The Tietze graph; (*b*) an embedding on the Möbius band



(a) (b)

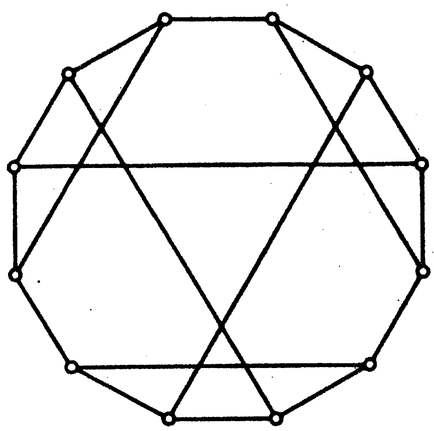
(*a*) The Petersen graph; (*b*) an embedding on the projective plane

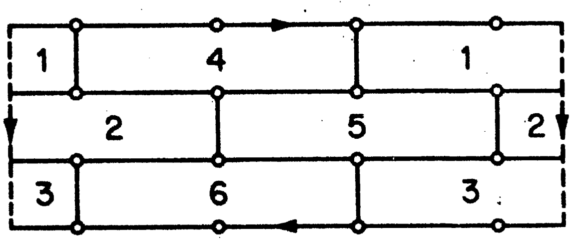




(*a*) The Heawood graph; (*b*) an embedding on the torus

Although the Klein bottle has characteristic 0, Franklin (1934) proved that it is only 6-chromatic, and found the following 6-chromatic map on the Klein bottle:





(*a*) The Franklin graph; (*b*) an embedding on the Klein bottle

It has been shown that, with the sole exception of the Klein bottle, equality holds in (III.2) for every surface S of characteristic $n<2$. This result is known as the *map colour theorem* (see Ringel, 1974).

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