Appendix IV
Unsolved Problems

Collected here are a number of unsolved problems of varying difficulty, with originators, dates and relevant bibliography. Conjectures are displayed in bold type. Problems marked † have now been solved; see page 253.

1. Two graphs G and H are *hypomorphic* (written $G \doteq H$) if there is a bijection $\sigma: V(G) \to V(H)$ such that $G - v \cong H - \sigma(v)$ for all $v \in V(G)$. A graph G is *reconstructible* if $G \doteq H$ implies $G \cong H$. The *reconstruction conjecture* claims that **every graph G with $\nu > 2$ is reconstructible** (S. M. Ulam, 1929). This has been verified for disconnected graphs, trees and a few other classes of graphs (see Harary, 1974).

   There is a corresponding *edge reconstruction conjecture:* **every graph G with $\varepsilon > 3$ is edge reconstructible.** Lovász (1972) has shown that every simple graph G with $\varepsilon > \binom{\nu}{2}\Big/2$ is edge reconstructible.

   P. K. Stockmeyer has found an infinite family of counterexamples to the analogous reconstruction conjecture for digraphs.

   Bondy, J. A. and Hemminger, R. L. (1976). Graph reconstruction—a survey. *J. Graph Theory*, to be published

   Lovász, L. (1972). A note on the line reconstruction problem. *J. Combinatorial Theory* B, **13,** 309–10

2. A graph G is *embeddable* in a graph H if G is isomorphic to a subgraph of H. Characterise the graphs embeddable in the k-cube (V. V. Firsov, 1965).

   Garey, M. R. and Graham, R. L. (1975). On cubical graphs. *J. Combinatorial Theory* (B), **18,** 84–95

3. **Every 4-regular simple graph contains a 3-regular subgraph** (N. Sauer, 1973).
4. **If $k > 2$, there exists no graph with the property that every pair of vertices is connected by a unique path of length k** (A. Kotzig, 1974). Kotzig has verified his conjecture for $k < 9$.
5. **Every connected graph G is the union of at most $[(\nu+1)/2]$ edge-disjoint paths** (T. Gallai, 1962). Lovász (1968) has shown that every graph G is the union of at most $[\nu/2]$ edge-disjoint paths and cycles.

Lovász, L. (1968). On coverings of graphs, in *Theory of Graphs* (eds. P. Erdös and G. Katona), Academic Press, New York, pp. 231–36

6. **Every 2-edge-connected simple graph G is the union of $\nu - 1$ cycles** (P. Erdös, A. W. Goodman and L. Pósa, 1966).

Erdös, P., Goodman, A. W. and Pósa, L. (1966). The representation of a graph by set intersections. *Canad. J. Math.*, **18,** 106–12

7. **If G is a simple block with at least $\nu/2 + k$ vertices of degree at least k, then G has a cycle of length at least $2k$** (D. R. Woodall, 1975).
8. Let $f(m, n)$ be the maximum possible number of edges in a simple graph on n vertices which contains no m-cycle. It is known that

$$f(m, n) = \begin{cases} [n^2/4] & \text{if} \quad m \quad \text{is odd}, \quad m \leq \frac{1}{2}(n+3) \\ \binom{n-m+2}{2} + \binom{m-1}{2} & \text{if} \quad m \geq \frac{1}{2}(n+3) \end{cases}$$

Determine $f(m, n)$ for the remaining cases (P. Erdös, 1963).

Bondy, J. A. and Simonovits, M. (1974). Cycles of even length in graphs. *J. Combinatorial Theory* (B), **16,** 97–105

Woodall, D. R. (1972). Sufficient conditions for circuits in graphs. *Proc. London Math. Soc.*, **24,** 739–55

9. Let $f(n)$ be the maximum possible number of edges in a simple graph on n vertices which contains no 3-regular subgraph. Determine $f(n)$ (P. Erdös and N. Sauer, 1974). Since there is a constant c such that every simple graph G with $\varepsilon \geq c\nu^{8/5}$ contains the 3-cube (Erdös and Simonovits, 1970), clearly $f(n) < cn^{8/5}$.

Erdös, P. and Simonovits, M. (1970). Some extremal problems in graph theory, in *Combinatorial Theory and its Applications I* (eds. P. Erdös, A. Rényi and V. T. Sós), North-Holland, Amsterdam, pp. 378–92

10. Determine which simple graphs G have exactly one cycle of each length l, $3 \leq l \leq \nu$ (R. C. Entringer, 1973).
11. Let $f(n)$ be the maximum possible number of edges in a graph on n vertices in which no two cycles have the same length. Determine $f(n)$ (P. Erdös, 1975).
12. **If G is simple and $\varepsilon > \nu(k-1)/2$, then G contains every tree with k edges** (P. Erdös and V. T. Sós, 1963). It is known that every such graph contains a path of length k (Erdös and Gallai, 1959).

Erdös, P. and Gallai, T. (1959). On maximal paths and circuits of graphs. *Acta Math. Acad. Sci. Hungar.*, **10,** 337–56

13. Find a (6, 5)-cage (see appendix III).

14. The *bandwidth* of G is defined to be

$$\min_{l} \max_{uv\in E} |l(u)-l(v)|$$

where the minimum is taken over all labellings l of V in distinct integers. Find bounds for the bandwidth of a graph (L. H. Harper, 1964). The bandwidth of the k-cube has been determined by Harper (1966).

Chvátalová, J. (1975). Optimal labelling of a product of two paths. *Discrete Math.*, **11**, 249–53

Harper, L. H. (1966). Optimal numberings and isoperimetric problems on graphs. *J. Combinatorial Theory*, **1**, 385–93

15. A simple graph G is *graceful* if there is a labelling l of its vertices with distinct integers from the set $\{0, 1, \ldots, \varepsilon\}$, so that the induced edge labelling l' defined by

$$l'(uv) = |l(u)-l(v)|$$

assigns each edge a different label. Characterise the graceful graphs (S. Golomb, 1972). It has been conjectured that, in particular, **every tree is graceful** (A. Rosa, 1966).

Golomb, S. (1972). How to number a graph, in *Graph Theory and Computing* (ed. R. C. Read), Academic Press, New York, pp. 23–37

†16. **The 3-connected planar graph on $2m$ edges with the least possible number of spanning trees is the wheel with m spokes** (W. T. Tutte, 1940).

Kelmans, A. K. and Chelnokov, V. M. (1974). A certain polynomial of a graph and graphs with an extremal number of trees. *J. Combinatorial Theory* (B), **16,** 197–214

17. Let u and v be two vertices in a graph G. Denote the minimum number of vertices whose deletion destroys all (u, v)-paths of length at most n by a_n, and the maximum number of internally disjoint (u, v)-paths of length at most n by b_n. Let $f(n)$ denote the maximum possible value of a_n/b_n. Determine $f(n)$ (V. Neumann, 1974). L. Lovász has conjectured that $\boldsymbol{f(n) \leq \sqrt{n}}$. It is known that

$$[\sqrt{n/2}] \leq f(n) \leq [n/2]$$

18. **Every 3-regular 3-connected bipartite planar graph is hamiltonian** (D. Barnette, 1970). P. Goodey has verified this conjecture for plane graphs whose faces are all of degree four or six. Note that if the planarity condition is dropped, the conjecture is no longer valid (see appendix III).

19. A graphic sequence $\mathbf{d}$ is *forcibly hamiltonian* if every simple graph with degree sequence $\mathbf{d}$ is hamiltonian. Characterise the forcibly hamiltonian

sequences (C. St. J. A. Nash-Williams, 1970). (Theorem 4.5 gives a partial solution.)

Nash-Williams, C. St. J. A. (1970). Valency sequences which force graphs to have Hamiltonian circuits: interim report, University of Waterloo preprint

20. **Every connected vertex-transitive graph has a Hamilton path** (L. Lovász, 1968). L. Babai has verified this conjecture for graphs with a prime number of vertices.

21. A graph G is *t-tough* if, for every vertex cut S, $\omega(G-S) \leq |S|/t$. (Thus theorem 4.2 says that every hamiltonian graph is 1-tough.)

(*a*) **If G is 2-tough, then G is hamiltonian** (V. Chvátal, 1971). C. Thomassen has obtained an example of a nonhamiltonian t-tough graph with $t > 3/2$

(*b*) **If G is 3/2-tough, then G has a 2-factor** (V. Chvátal, 1971).

Chvátal, V. (1973). Tough graphs and hamiltonian circuits. *Discrete Math.*, **5,** 215–28

22. The *binding number* of G is defined by

$$\text{bind } G = \min_{\substack{\emptyset \neq S \subseteq V \\ N(S) \neq V}} |N(S)|/|S|$$

(*a*) **If bind $G \geq 3/2$, then G contains a triangle** (D. R. Woodall, 1973).

(*b*) **If bind $G \geq 3/2$, then G is pancyclic** (contains cycles of all lengths l, $3 \leq l \leq \nu$) (D. R. Woodall, 1973).

Woodall (1973) has shown that G is hamiltonian if bind $G \geq 3/2$, and that G contains a triangle if bind $G \geq \frac{1}{2}(1+\sqrt{5})$.

Woodall, D. R. (1973). The binding number of a graph and its Anderson number. *J. Combinatorial Theory* (B), **15,** 225–55

23. **Every nonempty regular simple graph contains two disjoint maximal independent sets** (C. Payan, 1973)

24. Find the Ramsey number $r(3, 3, 3, 3)$. It is known that

$$51 \leq r(3, 3, 3, 3) \leq 65$$

Chung, F. R. K. (1973). On the Ramsey numbers $N(3, 3, \ldots, 3; 2)$, *Discrete Math.*, **5,** 317–21

Folkman, J. (1974). Notes on the Ramsey number $N(3, 3, 3, 3)$. *J. Combinatorial Theory* (A), **16,** 371–79

25. For $m < n$, let $f(m, n)$ denote the least possible number of vertices in a graph which contains no K_n but has the property that in every 2-edge colouring there is a monochromatic K_m. (Folkman, 1970 has established the existence of such graphs.) Determine bounds for $f(m, n)$. It is

known that

$$f(3, n) = 6 \quad \text{for} \quad n \geq 7$$
$$f(3, 6) = 8 \quad \text{(see exercise 7.2.5)}$$
$$10 \leq f(3, 5) \leq 18$$

Folkman, J. (1970). Graphs with monochromatic complete subgraphs in every edge coloring. *SIAM J. Appl. Math.*, **18,** 19–24

Irving, R. W. (1973). On a bound of Graham and Spencer for a graph-colouring constant. *J. Combinatorial Theory* (B), **15,** 200–203

Lin, S. On Ramsey numbers and K_r-coloring of graphs. *J. Combinatorial Theory* (B), **12,** 82–92

26. **If G is n-chromatic, then $r(G, G) \geq r(n, n)$** (P. Erdös, 1973). ($r(G, G)$ is defined in exercise 7.2.6.)
27. What is the maximum possible chromatic number of a graph which can be drawn in the plane so that each edge is a straight line segment of unit length? (L. Moser, 1958).

Erdös. P., Harary, F. and Tutte, W. T. (1965). On the dimension of a graph. *Mathematika*, **12,** 118–22

28. **The absolute values of the coefficients of any chromatic polynomial form a unimodal sequence** (that is, no term is flanked by terms of greater value) (R. C. Read, 1968).

Chvátal, V. (1970). A note on coefficients of chromatic polynomials. *J. Combinatorial Theory*, **9,** 95–96

29. **If G is not complete and $\chi = m + n - 1$, where $m \geq 2$ and $n \geq 2$, then there exist disjoint subgraphs G_1 and G_2 of G such that $\chi(G_1) = m$ and $\chi(G_2) = n$** (L. Lovász, 1968).
30. A simple graph G is *perfect* if, for every induced subgraph H of G, the number of vertices in a maximum clique is $\chi(H)$. **G is perfect if and only if no induced subgraph of G or G^c is an odd cycle of length greater than three** (C. Berge, 1961). This is the *strong perfect graph conjecture.* Lovász (1972) has shown that the complement of any perfect graph is perfect.

Lovász, L. (1972). Normal hypergraphs and the perfect graph conjecture. *Discrete Math.*, **2,** 253–67

Parthasarathy, K. R. and Ravindra, G. (to be published). The strong perfect-graph conjecture is true for $K_{1,3}$-free graphs. *J. Combinatorial Theory*

31. **If G is a 3-regular simple block and H is obtained from G by duplicating each edge, then $\chi'(H) = 6$** (D. R. Fulkerson, 1971).
32. **If G is simple, with ν even and $\chi'(G) = \Delta(G) + 1$, then $\chi'(G - v) = \chi'(G)$**

for some $v \in V$ (I. T. Jakobsen, L. W. Beineke and R. J. Wilson, 1973). This has been verified for all graphs G with $\nu \leq 10$ and all 3-regular graphs G with $\nu = 12$.

Beineke, L. W. and Wilson, R. J. (1973). On the edge-chromatic number of a graph. *Discrete Math.*, **5,** 15–20

33. **For any simple graph G, the elements of $V \cup E$ can be coloured in $\Delta + 2$ colours so that no two adjacent or incident elements receive the same colour** (M. Behzad, 1965). This is known as the *total colouring conjecture.* M. Rosenfeld and N. Vijayaditya have verified it for all graphs G with $\Delta \leq 3$.

Vijayaditya, N. (1971). On total chromatic number of a graph. *J. London Math. Soc.*, **3,** 405–408

34. **If G is simple and $\varepsilon > 3\nu - 6$, then G contains a subdivision of K_5** (G. A. Dirac, 1964). Thomassen (1975) has shown that G contains a subdivision of K_5 if $\varepsilon \geq 4\nu - 10$.

Dirac, G. A. (1964). Homomorphism theorems for graphs. *Math. Ann.*, **153,** 69–80

Thomassen, C. (1974). Some homeomorphism properties of graphs, *Math. Nachr.*, **64,** 119–33

35. A sequence **d** of non-negative integers is *potentially planar* if there is a simple planar graph with degree sequence **d**. Characterise the potentially planar sequences (S. L. Hakimi, 1963).

Owens, A. B. (1971). On the planarity of regular incidence sequences. *J. Combinatorial Theory* **(B), 11,** 201–12

†36. **If G is a loopless planar graph, then $\alpha \gtrapprox \nu/4$** (P. Erdös, 1968). Albertson (1974) has shown that every such graph satisfies $\alpha > 2\nu/9$.

Albertson, M. O. (1974). Finding an independent set in a planar graph, in *Graphs and Combinatorics* (eds. R. A. Bari and F. Harary), Springer-Verlag, New York, pp. 173–79

†37. **Every planar graph is 4-colourable** (F. Guthrie, 1852).

Ore, O. (1969). *The Four-Color Problem*, Academic Press, New York

38. **Every k-chromatic graph contains a subgraph contractible to K_k** (H. Hadwiger, 1943). Dirac (1964) has proved that every 6-chromatic graph contains a subgraph contractible to K_6 less one edge.

Dirac, G. A. (1964). Generalizations of the five colour theorem, in *Theory of Graphs and its Applications* (ed. M. Fiedler), Academic Press, New York, pp. 21–27

39. **Every k-chromatic graph contains a subdivision of K_k** (G. Hajós, 1961). Pelikán (1969) has shown that every 5-chromatic graph contains a subdivision of K_5 less one edge.

Pelikán, J. (1969). Valency conditions for the existence of certain subgraphs, in *Theory of Graphs* (eds. P. Erdös and G. Katona), Academic Press, New York, pp. 251–58

40. **Every 2-edge-connected 3-regular simple graph which has no Tait colouring contains a subgraph contractible to the Petersen graph** (W. T. Tutte, 1966).

    Isaacs, R. (1975). Infinite families of nontrivial trivalent graphs which are not Tait colourable. *Amer. Math. Monthly*, **82,** 221–39

    Tutte, W. T. (1966). On the algebraic theory of graph colorings. *J. Combinatorial Theory*, **1,** 15–50

41. **For every surface S, there exists a finite number of graphs which have minimum degree at least three and are minimally nonembeddable on S.**

†42. **If D is diconnected, then D has a directed cycle of length at least χ** (M. Las Vergnas, 1974).

43. **If D is a tournament with ν odd and every indegree and outdegree equal to $(\nu-1)/2$, then D is the union of $(\nu-1)/2$ arc-disjoint directed Hamilton cycles** (P. Kelly, 1966).

44. **If D is a tournament with ν even, then D is the union of $\sum_{v\in V} \max\{0, d^+(v)-d^-(v)\}$ arc-disjoint directed paths** (R. O'Brien, 1974). This would imply the truth of conjecture 43.

45. Characterise the tournaments D with the property that all subtournaments $D-v$ are isomorphic (A. Kotzig, 1973).

46. **If D is a digraph which contains a directed cycle, then there is some arc whose reversal decreases the number of directed cycles in D** (A. Adám, 1963).

47. **Given a positive integer n, there exists a least integer $f(n)$ such that in any digraph with at most n arc-disjoint directed cycles there are $f(n)$ arcs whose deletion destroys all directed cycles** (T. Gallai, 1964; D. H. Younger, 1968).

    Erdös. P. and Pósa, L. (1962). On the maximal number of disjoint circuits of a graph. *Publ. Math. Debrecen*, **9,** 3–12

    Younger, D. H. (1973). Graphs with interlinked directed circuits, in *Proceedings of Midwest Symposium on Circuit Theory*

48. An $(m+n)$-regular graph is (m, n)-*orientable* if it can be oriented so that each indegree is either m or n. **Every 5-regular simple graph with no 1-edge cut or 3-edge cut is (4, 1)-orientable** (W. T. Tutte, 1972). Tutte has shown that this would imply Grötzsch's theorem

49. Obtain an algorithm to find a maximum flow in a network with two sources x_1 and x_2, two sinks y_1 and y_2, and two commodities, the requirement being to ship commodity 1 from x_1 to y_1 and commodity 2 from x_2 to y_2 (L. R. Ford and D. R. Fulkerson, 1962).

Rothschild, B. and Whinston, A. (1966). On two commodity network flows. *Operations Res.*, **14,** 377–87

50. **Every 2-edge-connected digraph D has a circulation f over the field of integers modulo 5 in which $f(a) \neq 0$ for all arcs a** (W. T. Tutte, 1949). Tutte has shown that this would imply the five-colour theorem.

References for problems solved since first printing:

16. Göbel, F. and Jagers, A. A. (1976). On a conjecture of Tutte concerning minimal tree numbers. *J. Combinatorial Theory* (B), to be published

36 and 37. Appel, K. and Haken, W. (1976). Every planar map is four colorable. *Bull. Amer. Math. Soc.*, **82,** 711–2

42. Bondy, J. A. (1976). Diconnected orientations and a conjecture of Las Vergnas. *J. London Math. Soc.*, to be published