3 Connectivity

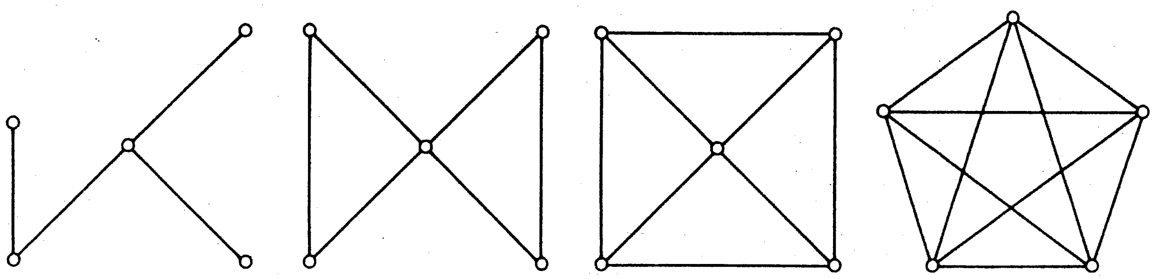
3.1 CONNECTIVITY

In section 1.6 we introduced the concept of connection in graphs. Consider, now, the four connected graphs of figure 3.1.

G_1 is a tree, a minimal connected graph; deleting any edge disconnects it. G_2 cannot be disconnected by the deletion of a single edge, but can be disconnected by the deletion of one vertex, its cut vertex. There are no cut edges or cut vertices in G_3, but even so G_3 is clearly not as well connected as G_4, the complete graph on five vertices. Thus, intuitively, each successive graph is more strongly connected than the previous one. We shall now define two parameters of a graph, its connectivity and edge connectivity, which measure the extent to which it is connected.

A *vertex cut* of G is a subset V' of V such that $G-V'$ is disconnected. A k-*vertex cut* is a vertex cut of k elements. A complete graph has no vertex cut; in fact, the only graphs which do not have vertex cuts are those that contain complete graphs as spanning subgraphs. If G has at least one pair of distinct nonadjacent vertices, the *connectivity* $\kappa(G)$ of G is the minimum k for which G has a k-vertex cut; otherwise, we define $\kappa(G)$ to be $\nu-1$. Thus $\kappa(G)=0$ if G is either trivial or disconnected. G is said to be k-*connected* if $\kappa(G)\geq k$. All nontrivial connected graphs are 1-connected.

Recall that an edge cut of G is a subset of E of the form $[S, \bar{S}]$, where S is a nonempty proper subset of V. A k-*edge cut* is an edge cut of k elements. If G is nontrivial and E' is an edge cut of G, then $G-E'$ is disconnected; we then define the *edge connectivity* $\kappa'(G)$ of G to be the minimum k for which G has a k-edge cut. If G is trivial, $\kappa'(G)$ is defined to be zero. Thus $\kappa'(G)=0$ if G is either trivial or disconnected, and $\kappa'(G)=1$ if G is a connected graph with a cut edge. G is said to be k-*edge-connected* if $\kappa'(G)\geq k$. All nontrivial connected graphs are 1-edge-connected.



G_1 G_2 G_3 G_4

Figure 3.1

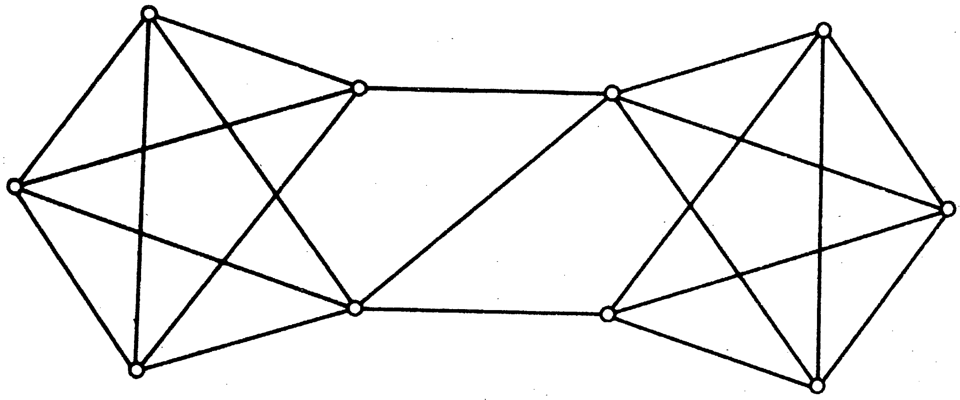


Figure 3.2

Theorem 3.1 $\kappa \leq \kappa' \leq \delta$.

Proof If G is trivial, then $\kappa' = 0 \leq \delta$. Otherwise, the set of links incident with a vertex of degree δ constitute a δ-edge cut of G. It follows that $\kappa' \leq \delta$.

We prove that $\kappa \leq \kappa'$ by induction on κ'. The result is true if $\kappa' = 0$, since then G must be either trivial or disconnected. Suppose that it holds for all graphs with edge connectivity less than k, let G be a graph with $\kappa'(G) = k > 0$, and let e be an edge in a k-edge cut of G. Setting $H = G - e$, we have $\kappa'(H) = k - 1$ and so, by the induction hypothesis, $\kappa(H) \leq k - 1$.

If H contains a complete graph as a spanning subgraph, then so does G and

$$\kappa(G) = \kappa(H) \leq k - 1$$

Otherwise, let S be a vertex cut of H with $\kappa(H)$ elements. Since $H - S$ is disconnected, either $G - S$ is disconnected, and then

$$\kappa(G) \leq \kappa(H) \leq k - 1$$

or else $G - S$ is connected and e is a cut edge of $G - S$. In this latter case, either $\nu(G - S) = 2$ and

$$\kappa(G) \leq \nu(G) - 1 = \kappa(H) + 1 \leq k$$

or (exercise 2.3.1a) $G - S$ has a 1-vertex cut $\{v\}$, implying that $S \cup \{v\}$ is a vertex cut of G and

$$\kappa(G) \leq \kappa(H) + 1 \leq k$$

Thus in each case we have $\kappa(G) \leq k = \kappa'(G)$. The result follows by the principle of induction □

The inequalities in theorem 3.1 are often strict. For example, the graph G of figure 3.2 has $\kappa = 2$, $\kappa' = 3$ and $\delta = 4$.

Exercises

3.1.1 (*a*) Show that if G is k-edge-connected, with $k>0$, and if E' is a set of k edges of G, then $\omega(G-E')\leq 2$.

(*b*) For $k>0$, find a k-connected graph G and a set V' of k vertices of G such that $\omega(G-V')>2$.

3.1.2 Show that if G is k-edge-connected, then $\varepsilon\geq k\nu/2$.

3.1.3 (*a*) Show that if G is simple and $\delta\geq\nu-2$, then $\kappa=\delta$.

(*b*) Find a simple graph G with $\delta=\nu-3$ and $\kappa<\delta$.

3.1.4 (*a*) Show that if G is simple and $\delta\geq\nu/2$, then $\kappa'=\delta$.

(*b*) Find a simple graph G with $\delta=[(\nu/2)-1]$ and $\kappa'<\delta$.

3.1.5 Show that if G is simple and $\delta\geq(\nu+k-2)/2$, then G is k-connected.

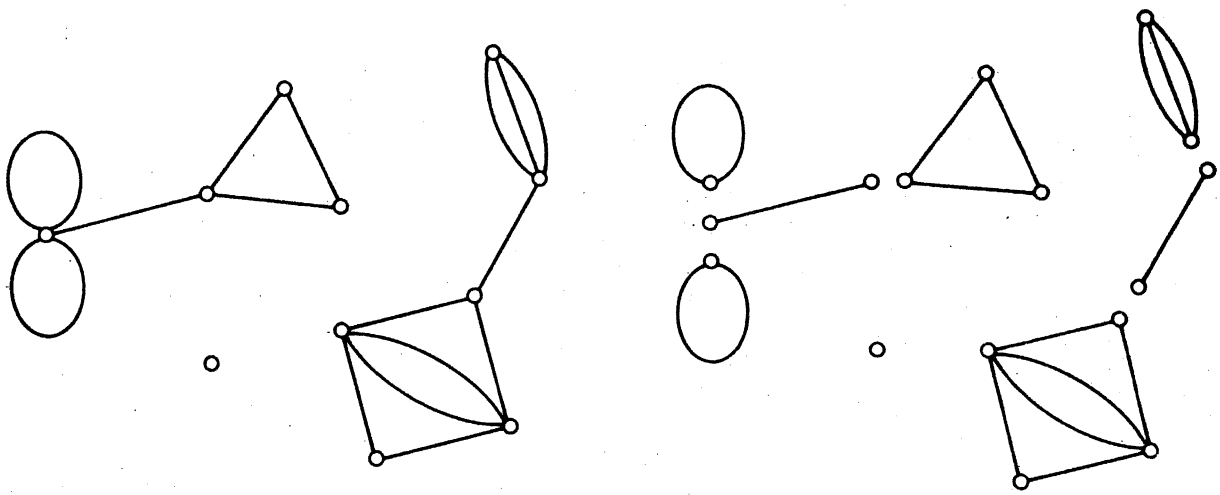
3.1.6 Show that if G is simple and 3-regular, then $\kappa=\kappa'$.

3.1.7 Show that if l, m and n are integers such that $0<l\leq m\leq n$, then there exists a simple graph G with $\kappa=l$, $\kappa'=m$, and $\delta=n$.

(G. Chartrand and F. Harary)

3.2 BLOCKS

A connected graph that has no cut vertices is called a *block*. Every block with at least three vertices is 2-connected. A *block of a graph* is a subgraph that is a block and is maximal with respect to this property. Every graph is the union of its blocks; this is illustrated in figure 3.3.



(a) (b)

Figure 3.3. (*a*) G; (*b*) the blocks of G

A family of paths in G is said to be *internally-disjoint* if no vertex of G is an internal vertex of more than one path of the family. The following theorem is due to Whitney (1932).

Theorem 3.2 A graph G with $\nu\geq 3$ is 2-connected if and only if any two vertices of G are connected by at least two internally-disjoint paths.

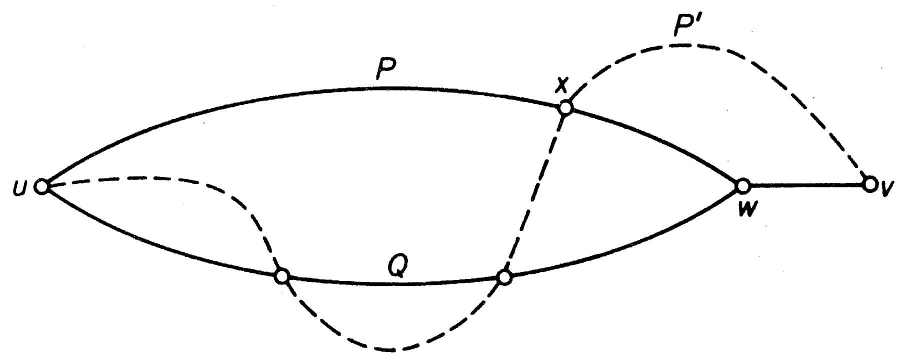


Figure 3.4

Proof If any two vertices of G are connected by at least two internally-disjoint paths then, clearly, G is connected and has no 1-vertex cut. Hence G is 2-connected.

Conversely, let G be a 2-connected graph. We shall prove, by induction on the distance $d(u, v)$ between u and v, that any two vertices u and v are connected by at least two internally-disjoint paths.

Suppose, first, that $d(u, v) = 1$. Then, since G is 2-connected, the edge uv is not a cut edge and therefore, by theorem 2.3, it is contained in a cycle. It follows that u and v are connected by two internally-disjoint paths in G.

Now assume that the theorem holds for any two vertices at distance less than k, and let $d(u, v) = k \geq 2$. Consider a (u, v)-path of length k, and let w be the vertex that precedes v on this path. Since $d(u, w) = k - 1$, it follows from the induction hypothesis that there are two internally-disjoint (u, w)-paths P and Q in G. Also, since G is 2-connected, $G - w$ is connected and so contains a (u, v)-path P'. Let x be the last vertex of P' that is also in $P \cup Q$ (see figure 3.4). Since u is in $P \cup Q$, there is such an x; we do not exclude the possibility that $x = v$.

We may assume, without loss of generality, that x is in P. Then G has two internally-disjoint (u, v)-paths, one composed of the section of P from u to x together with the section of P' from x to v, and the other composed of Q together with the path wv □

Corollary 3.2.1 If G is 2-connected, then any two vertices of G lie on a common cycle.

Proof This follows immediately from theorem 3.2 since two vertices lie on a common cycle if and only if they are connected by two internally-disjoint paths □

It is convenient, now, to introduce the operation of subdivision of an edge. An edge e is said to be *subdivided* when it is deleted and replaced by a path of length two connecting its ends, the internal vertex of this path being a new vertex. This is illustrated in figure 3.5.

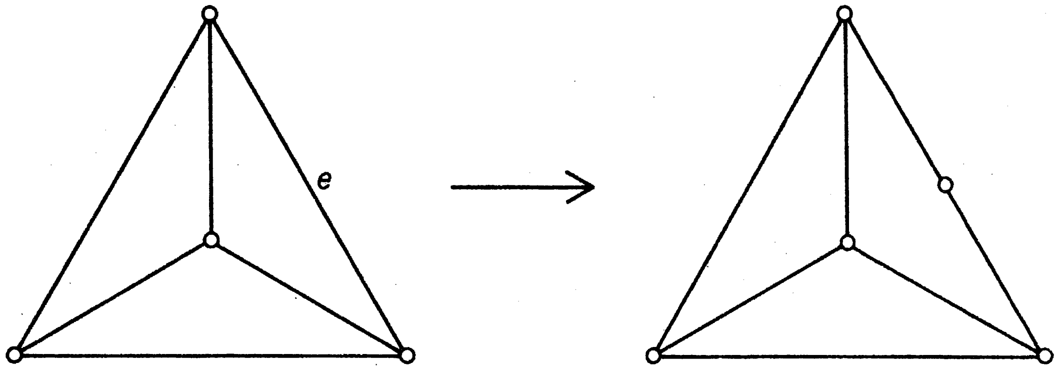


Figure 3.5. Subdivision of an edge

It can be seen that the class of blocks with at least three vertices is closed under the operation of subdivision. The proof of the next corollary uses this fact.

Corollary 3.2.2 If G is a block with $\nu \geq 3$, then any two edges of G lie on a common cycle.

Proof Let G be a block with $\nu \geq 3$, and let e_1 and e_2 be two edges of G. Form a new graph G' by subdividing e_1 and e_2, and denote the new vertices by v_1 and v_2. Clearly, G' is a block with at least five vertices, and hence is 2-connected. It follows from corollary 3.2.1 that v_1 and v_2 lie on a common cycle of G'. Thus e_1 and e_2 lie on a common cycle of G (see figure 3.6) □

Theorem 3.2 has a generalisation to k-connected graphs, known as *Menger's theorem*: a graph G with $\nu \geq k+1$ is k-connected if and only if any two distinct vertices of G are connected by at least k internally-disjoint paths. There is also an edge analogue of this theorem: a graph G is k-edge-connected if and only if any two distinct vertices of G are connected

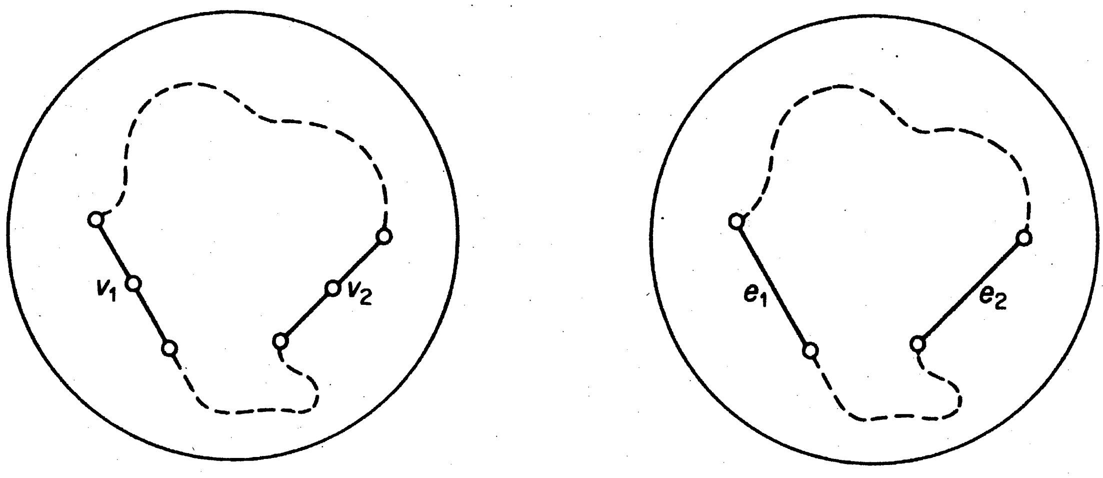


Figure 3.6. (*a*) G'; (*b*) G

by at least k edge-disjoint paths. Proofs of these theorems will be given in chapter 11.

Exercises

3.2.1 Show that a graph is 2-edge-connected if and only if any two vertices are connected by at least two edge-disjoint paths.

3.2.2 Give an example to show that if P is a (u, v)-path in a 2-connected graph G, then G does not necessarily contain a (u, v)-path Q internally-disjoint from P.

3.2.3 Show that if G has no even cycles, then each block of G is either K_1 or K_2, or an odd cycle.

3.2.4 Show that a connected graph which is not a block has at least two blocks that each contain exactly one cut vertex.

3.2.5 Show that the number of blocks in G is equal to $\omega + \sum_{v \in V} (b(v) - 1)$, where $b(v)$ denotes the number of blocks of G containing v.

3.2.6* Let G be a 2-connected graph and let X and Y be disjoint subsets of V, each containing at least two vertices. Show that G contains disjoint paths P and Q such that

(i) the origins of P and Q belong to X,
(ii) the termini of P and Q belong to Y, and
(iii) no internal vertex of P or Q belongs to $X \cup Y$.

3.2.7* A nonempty graph G is *κ-critical* if, for every edge e, $\kappa(G-e) < \kappa(G)$.

(a) Show that every κ-critical 2-connected graph has a vertex of degree two.
(Halin, 1969 has shown that, in general, every κ-critical k-connected graph has a vertex of degree k.)

(b) Show that if G is a κ-critical 2-connected graph with $\nu \geq 4$, then $\varepsilon \leq 2\nu - 4$. (G. A. Dirac)

3.2.8 Describe a good algorithm for finding the blocks of a graph.

APPLICATIONS

3.3 CONSTRUCTION OF RELIABLE COMMUNICATION NETWORKS

If we think of a graph as representing a communication network, the connectivity (or edge connectivity) becomes the smallest number of communication stations (or communication links) whose breakdown would jeopardise communication in the system. The higher the connectivity and edge connectivity, the more reliable the network. From this point of view, a tree network, such as the one obtained by Kruskal's algorithm, is not very reliable, and one is led to consider the following generalisation of the connector problem.

Let k be a given positive integer and let G be a weighted graph. Determine a minimum-weight k-connected spanning subgraph of G.

For $k=1$, this problem reduces to the connector problem, which can be solved by Kruskal's algorithm. For values of k greater than one, the problem is unsolved and is known to be difficult. However, if G is a complete graph in which each edge is assigned unit weight, then the problem has a simple solution which we now present.

Observe that, for a weighted complete graph on n vertices in which each edge is assigned unit weight, a minimum-weight m-connected spanning subgraph is simply an m-connected graph on n vertices with as few edges as possible. We shall denote by $f(m, n)$ the least number of edges that an m-connected graph on n vertices can have. (It is, of course, assumed that $m<n$.) By theorems 3.1 and 1.1

$$f(m, n) \geq \{mn/2\} \tag{3.1}$$

We shall show that equality holds in (3.1) by constructing an m-connected graph $H_{m,n}$ on n vertices that has exactly $\{mn/2\}$ edges. The structure of $H_{m,n}$ depends on the parities of m and n; there are three cases.

Case 1 m even. Let $m=2r$. Then $H_{2r,n}$ is constructed as follows. It has vertices $0, 1, \ldots, n-1$ and two vertices i and j are joined if $i-r \leq j \leq i+r$ (where addition is taken modulo n). $H_{4,8}$ is shown in figure 3.7*a*.

Case 2 m odd, n even. Let $m=2r+1$. Then $H_{2r+1,n}$ is constructed by first drawing $H_{2r,n}$ and then adding edges joining vertex i to vertex $i+(n/2)$ for $1 \leq i \leq n/2$. $H_{5,8}$ is shown in figure 3.7*b*.

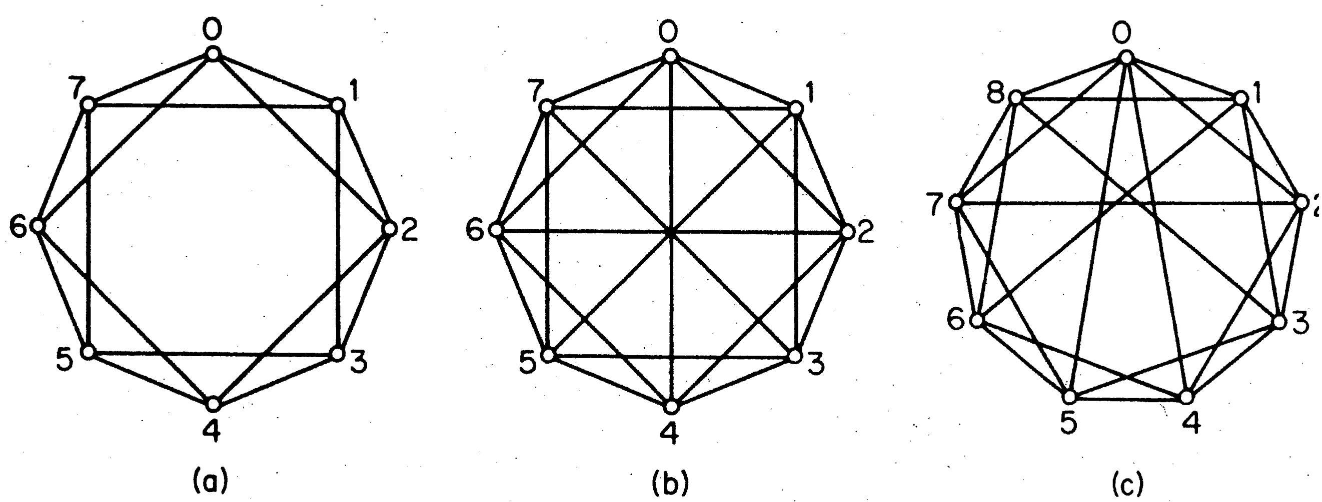


Figure 3.7. (*a*) $H_{4,8}$; (*b*) $H_{5,8}$; (*c*) $H_{5,9}$

Case 3 m odd, n odd. Let $m = 2r+1$. Then $H_{2r+1,n}$ is constructed by first drawing $H_{2r,n}$ and then adding edges joining vertex 0 to vertices $(n-1)/2$ and $(n+1)/2$ and vertex i to vertex $i+(n+1)/2$ for $1 \leq i < (n-1)/2$. $H_{5,9}$ is shown in figure 3.7*c*.

Theorem 3.3 (Harary, 1962) The graph $H_{m,n}$ is m-connected.

Proof Consider the case $m = 2r$. We shall show that $H_{2r,n}$ has no vertex cut of fewer than $2r$ vertices. If possible, let V' be a vertex cut with $|V'| < 2r$. Let i and j be vertices belonging to different components of $H_{2r,n} - V'$. Consider the two sets of vertices

$$S = \{i, i+1, \ldots, j-1, j\}$$

and

$$T = \{j, j+1, \ldots, i-1, i\}$$

where addition is taken modulo n. Since $|V'| < 2r$, we may assume, without loss of generality, that $|V' \cap S| < r$. Then there is clearly a sequence of distinct vertices in $S \setminus V'$ which starts with i, ends with j, and is such that the difference between any two consecutive terms is at most r. But such a sequence is an (i, j)-path in $H_{2r,n} - V'$, a contradiction. Hence $H_{2r,n}$ is $2r$-connected.

The case $m = 2r+1$ is left as an exercise (exercise 3.3.1) □

It is easy to see that $\varepsilon(H_{m,n}) = \{mn/2\}$. Thus, by theorem 3.3,

$$f(m, n) \leq \{mn/2\} \tag{3.2}$$

It now follows from (3.1) and (3.2) that

$$f(m, n) = \{mn/2\}$$

and that $H_{m,n}$ is an m-connected graph on n vertices with as few edges as possible.

We note that since, for any graph G, $\kappa \leq \kappa'$ (theorem 3.1), $H_{m,n}$ is also m-edge-connected. Thus, denoting by $g(m, n)$ the least possible number of edges in an m-edge-connected graph on n vertices, we have, for $1 < m < n$

$$g(m, n) = \{mn/2\} \tag{3.3}$$

Exercises

3.3.1 Show that $H_{2r+1,n}$ is $(2r+1)$-connected.

3.3.2 Show that $\kappa(H_{m,n}) = \kappa'(H_{m,n}) = m$.

3.3.3 Find a graph with nine vertices and 23 edges that is 5-connected but not isomorphic to the graph $H_{5,9}$ of figure 3.7*c*.

3.3.4 Show that (3.3) holds for all values of m and n with $m > 1$ and $n > 1$.

3.3.5 Find, for all $\nu \geq 5$, a 2-connected graph G of diameter two with $\varepsilon = 2\nu - 5$.

(Murty, 1969 has shown that every such graph has at least this number of edges.)

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