6 Edge Colourings

6.1 EDGE CHROMATIC NUMBER

A *k-edge colouring* $\mathscr{C}$ of a loopless graph G is an assignment of k colours, $1, 2, \ldots, k$, to the edges of G. The colouring $\mathscr{C}$ is *proper* if no two adjacent edges have the same colour.

Alternatively, a k-edge colouring can be thought of as a partition $(E_1, E_2, \ldots, E_k)$ of E, where E denotes the (possibly empty) subset of E assigned colour i. A proper k-edge colouring is then a k-edge colouring $(E_1, E_2, \ldots, E_k)$ in which each subset E_i is a matching. The graph of figure 6.1 has the proper 4-edge colouring $(\{a, g\}, \{b, e\}, \{c, f\}, \{d\})$.

G is *k-edge colourable* if G has a proper k-edge-colouring. Trivially, every loopless graph G is ε-edge-colourable; and if G is k-edge-colourable, then G is also l-edge-colourable for every $l > k$. The ***edge chromatic number*** $\chi'(G)$, of a loopless graph G, is the minimum k for which G is k-edge-colourable. G is *k-edge-chromatic* if $\chi'(G) = k$. It can be readily verified that the graph of figure 6.1 has no proper 3-edge colouring. This graph is therefore 4-edge-chromatic.

Clearly, in any proper edge colouring, the edges incident with any one vertex must be assigned different colours. It follows that

$$\chi' \geq \Delta \tag{6.1}$$

Referring to the example of figure 6.1, we see that inequality (6.1) may be strict. However, we shall show that, in the case when G is bipartite, $\chi' = \Delta$. The following simple lemma is basic to our proof. We say that colour i is *represented* at vertex v if some edge incident with v has colour i.

Lemma 6.1.1 Let G be a connected graph that is not an odd cycle. Then

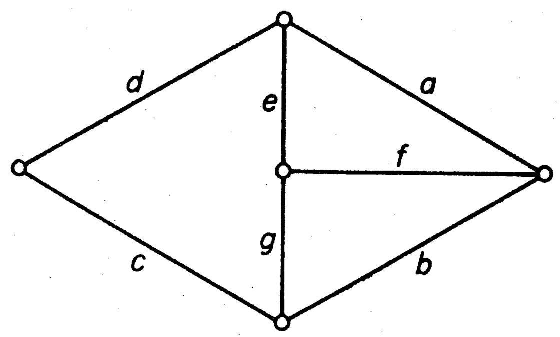


Figure 6.1

G has a 2-edge colouring in which both colours are represented at each vertex of degree at least two.

Proof We may clearly assume that G is nontrivial. Suppose, first, that G is eulerian. If G is an even cycle, the proper 2-edge colouring of G has the required property. Otherwise, G has a vertex v_0 of degree at least four. Let $v_0e_1v_1 \ldots e_\varepsilon v_0$ be an Euler tour of G, and set

$$E_1 = \{e_i \mid i \text{ odd}\} \quad \text{and} \quad E_2 = \{e_i \mid i \text{ even}\} \tag{6.2}$$

Then the 2-edge colouring (E_1, E_2) of G has the required property, since each vertex of G is an internal vertex of $v_0e_1v_1 \ldots e_\varepsilon v_0$.

If G is not eulerian, construct a new graph G^* by adding a new vertex v_0 and joining it to each vertex of odd degree in G. Clearly G^* is eulerian. Let $v_0e_1v_1 \ldots e_{\varepsilon^*} v_0$ be an Euler tour of G^* and define E_1 and E_2 as in (6.2). It is then easily verified that the 2-edge colouring $(E_1 \cap E, E_2 \cap E)$ of G has the required property □

Given a k-edge colouring $\mathscr{C}$ of G we shall denote by $c(v)$ the number of distinct colours represented at v. Clearly, we always have

$$c(v) \leq d(v) \tag{6.3}$$

Moreover, $\mathscr{C}$ is a proper k-edge colouring if and only if equality holds in (6.3) for all vertices v of G. We shall call a k-edge colouring $\mathscr{C}'$ an *improvement* on $\mathscr{C}$ if

$$\sum_{v \in V} c'(v) > \sum_{v \in V} c(v)$$

where $c'(v)$ is the number of distinct colours represented at v in the colouring $\mathscr{C}'$. An *optimal* k-edge colouring is one which cannot be improved.

Lemma 6.1.2 Let $\mathscr{C} = (E_1, E_2, \ldots, E_k)$ be an optimal k-edge colouring of G. If there is a vertex u in G and colours i and j such that i is not represented at u and j is represented at least twice at u, then the component of $G[E_i \cup E_j]$ that contains u is an odd cycle.

Proof Let u be a vertex that satisfies the hypothesis of the lemma, and denote by H the component of $G[E_i \cup E_j]$ containing u. Suppose that H is not an odd cycle. Then, by lemma 6.1.1, H has a 2-edge colouring in which both colours are represented at each vertex of degree at least two in H. When we recolour the edges of H with colours i and j in this way, we obtain a new k-edge colouring $\mathscr{C}' = (E_1', E_2', \ldots, E_k')$ of G. Denoting by $c'(v)$ the number of distinct colours at v in the colouring $\mathscr{C}'$, we have

$$c'(u) = c(u) + 1$$

since, now, both i and j are represented at u, and also

$$c'(v) \geq c(v) \quad \text{for} \quad v \neq u$$

Thus $\sum_{v \in V} c'(v) > \sum_{v \in V} c(v)$, contradicting the choice of $\mathscr{C}$. It follows that H is indeed an odd cycle □

Theorem 6.1 If G is bipartite, then $\chi' = \Delta$.

Proof Let G be a graph with $\chi' > \Delta$, let $\mathscr{C} = (E_1, E_2, \ldots, E_\Delta)$ be an optimal Δ-edge colouring of G, and let u be a vertex such that $c(u) < d(u)$. Clearly, u satisfies the hypothesis of lemma 6.1.2. Therefore G contains an odd cycle and so is not bipartite. It follows from (6.1) that if G is bipartite, then $\chi' = \Delta$ □

An alternative proof of theorem 6.1, using exercise 5.2.3a, is outlined in exercise 6.1.3.

Exercises

6.1.1 Show, by finding an appropriate edge colouring, that $\chi'(K_{m,n}) = \Delta(K_{m,n})$.

6.1.2 Show that the Petersen graph is 4-edge-chromatic.

6.1.3 (*a*) Show that if G is bipartite, then G has a Δ-regular bipartite supergraph.
(*b*) Using (*a*) and exercise 5.2.3a, give an alternative proof of theorem 6.1.

6.1.4 Describe a good algorithm for finding a proper Δ-edge colouring of a bipartite graph G.

6.1.5 Using exercise 1.5.8 and theorem 6.1, show that if G is loopless with $\Delta = 3$, then $\chi' \leq 4$.

6.1.6 Show that if G is bipartite with $\delta > 0$, then G has a δ-edge colouring such that all δ colours are represented at each vertex.
(R. P. Gupta)

6.2 VIZING'S THEOREM

As has already been noted, if G is not bipartite then we cannot necessarily conclude that $\chi' = \Delta$. An important theorem due to Vizing (1964) and, independently, Gupta (1966), asserts that, for any simple graph G, either $\chi' = \Delta$ or $\chi' = \Delta + 1$. The proof given here is by Fournier (1973).

Theorem 6.2 If G is simple, then either $\chi' = \Delta$ or $\chi' = \Delta + 1$.

Proof Let G be a simple graph. By virtue of (6.1) we need only show that $\chi' \leq \Delta + 1$. Suppose, then, that $\chi' > \Delta + 1$. Let $\mathscr{C} = (E_1, E_2, \ldots, E_{\Delta+1})$ be

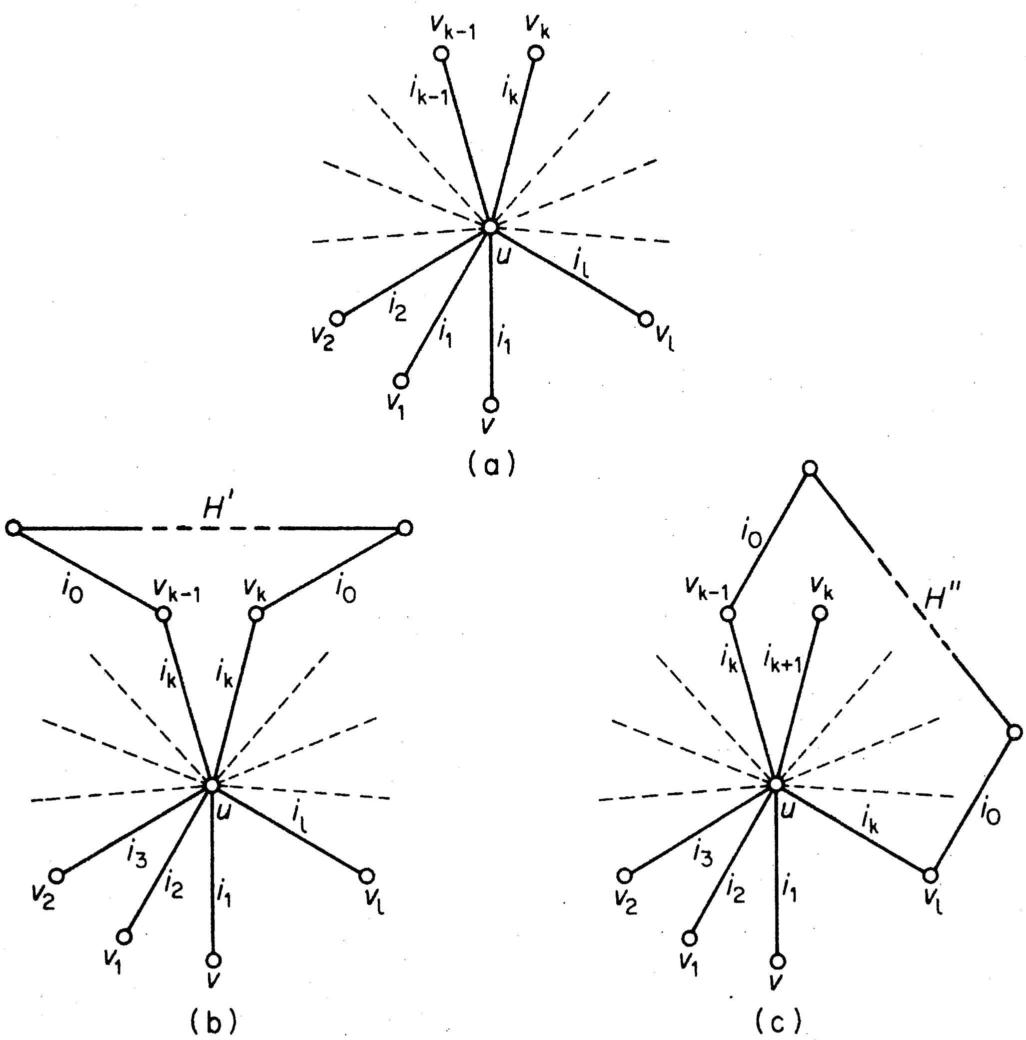


Figure 6.2

an optimal $(\Delta+1)$-edge colouring of G and let u be a vertex such that $c(u)<d(u)$. Then there exist colours i_0 and i_1 such that i_0 is not represented at u, and i_1 is represented at least twice at u. Let uv_1 have colour i_1, as in figure 6.2a.

Since $d(v_1)<\Delta+1$, some colour i_2 is not represented at v_1. Now i_2 must be represented at u since otherwise, by recolouring uv_1 with i_2, we would obtain an improvement on $\mathscr{C}$. Thus some edge uv_2 has colour i_2. Again, since $d(v_2)<\Delta+1$, some colour i_3 is not represented at v_2; and i_3 must be represented at u since otherwise, by recolouring uv_1 with i_2 and uv_2 with i_3, we would obtain an improved $(\Delta+1)$-edge colouring. Thus some edge uv_3 has colour i_3. Continuing this procedure we construct a sequence $v_1, v_2, \ldots$ of vertices and a sequence $i_1, i_2, \ldots$ of colours, such that

(i) uv_j has colour i_j, and
(ii) i_{j+1} is not represented at v_j.

Since the degree of u is finite, there exists a smallest integer l such that, for some $k<l$,

(iii) $i_{l+1}=i_k$.

The situation is depicted in figure 6.2*a*.

We now recolour G as follows. For $1 \leq j \leq k-1$, recolour uv_j with colour i_{j+1}, yielding a new $(\Delta+1)$-edge colouring $\mathscr{C}' = (E'_1, E'_2, \ldots, E'_{\Delta+1})$ (figure 6.2*b*). Clearly

$$c'(v) \geq c(v) \quad \text{for all} \quad v \in V$$

and therefore $\mathscr{C}'$ is also an optimal $(\Delta+1)$-edge colouring of G. By lemma 6.1.2, the component H' of $G[E'_{i_0} \cup E'_{i_k}]$ that contains u is an odd cycle.

Now, in addition, recolour uv_j with colour i_{j+1}, $k \leq j \leq l-1$, and uv_l with colour i_k, to obtain a $(\Delta+1)$-edge colouring $\mathscr{C}'' = (E''_1, E''_2, \ldots, E''_{\Delta+1})$ (figure 6.2*c*). As above

$$c''(v) \geq c(v) \quad \text{for all} \quad v \in V$$

and the component H'' of $G[E''_{i_0} \cup E''_{i_k}]$ that contains u is an odd cycle. But, since v_k has degree two in H', v_k clearly has degree one in H''. This contradiction establishes the theorem □

Actually, Vizing proved a more general theorem than that given above, one that is valid for all loopless graphs. The maximum number of edges joining two vertices in G is called the *multiplicity* of G, and denoted by $\mu(G)$. We can now state Vizing's theorem in its full generality: if G is loopless, then $\Delta \leq \chi' \leq \Delta + \mu$.

This theorem is best possible in the sense that, for any μ, there exists a graph G such that $\chi' = \Delta + \mu$. For example, in the graph G of figure 6.3, $\Delta = 2\mu$ and, since any two edges are adjacent, $\chi' = \varepsilon = 3\mu$.

Strong as theorem 6.2 is, it leaves open one interesting question: which simple graphs satisfy $\chi' = \Delta$? The significance of this question will become apparent in chapter 9, when we study edge colourings of planar graphs.

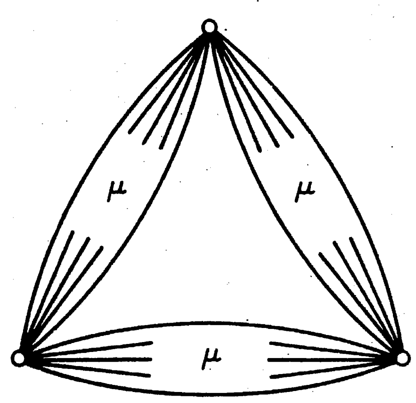


Figure 6.3. A graph G with $\chi' = \Delta + \mu$

Exercises

6.2.1* Show, by finding appropriate edge colourings, that $\chi'(K_{2n-1}) = \chi'(K_{2n}) = 2n-1$.

6.2.2 Show that if G is a nonempty regular simple graph with ν odd, then $\chi' = \Delta+1$.

6.2.3 (*a*) Let G be a simple graph. Show that if $\nu = 2n+1$ and $\varepsilon > n\Delta$, then $\chi' = \Delta+1$. (V. G. Vizing)

(*b*) Using (*a*), show that

(i) if G is obtained from a simple regular graph with an even number of vertices by subdividing one edge, then $\chi' = \Delta+1$;

(ii) if G is obtained from a simple k-regular graph with an odd number of vertices by deleting fewer than $k/2$ edges, then $\chi' = \Delta+1$. (L. W. Beineke and R. J. Wilson)

6.2.4 (*a*) Show that if G is loopless, then G has a Δ-regular loopless supergraph.

(*b*) Using (*a*) and exercise 5.2.3*b*, show that if G is loopless and Δ is even, then $\chi' \leq 3\Delta/2$.

(Shannon, 1949 has shown that this inequality also holds when Δ is odd.)

6.2.5 G is called *uniquely k-edge-colourable* if any two proper k-edge colourings of G induce the same partition of E. Show that every uniquely 3-edge-colourable 3-regular graph is hamiltonian. (D. L. Greenwell and H. V. Kronk)

6.2.6 The *product* of simple graphs G and H is the simple graph $G \times H$ with vertex set $V(G) \times V(H)$, in which (u, v) is adjacent to (u', v') if and only if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$.

(*a*) Using Vizing's theorem (6.2), show that $\chi'(G \times K_2) = \Delta(G \times K_2)$.

(*b*) Deduce that if H is nontrivial with $\chi'(H) = \Delta(H)$, then $\chi'(G \times H) = \Delta(G \times H)$.

6.2.7 Describe a good algorithm for finding a proper $(\Delta+1)$-edge colouring of a simple graph G.

6.2.8* Show that if G is simple with $\delta > 1$, then G has a $(\delta-1)$-edge colouring such that all $\delta-1$ colours are represented at each vertex. (R. P. Gupta)

APPLICATIONS

6.3 THE TIMETABLING PROBLEM

In a school, there are m teachers $X_1, X_2, \ldots, X_m$, and n classes $Y_1, Y_2, \ldots, Y_n$. Given that teacher X_i is required to teach class Y_j for p_{ij} periods, schedule a complete timetable in the minimum possible number of periods.

The above problem is known as the *timetabling problem*, and can be solved completely using the theory of edge colourings developed in this chapter. We represent the teaching requirements by a bipartite graph G with bipartition (X, Y), where $X=\{x_1, x_2, \ldots, x_m\}$, $Y=\{y_1, y_2, \ldots, y_n\}$ and vertices x_i and y_j are joined by p_{ij} edges. Now, in any one period, each teacher can teach at most one class, and each class can be taught by at most one teacher—this, at least, is our assumption. Thus a teaching schedule for one period corresponds to a matching in the graph and, conversely, each matching corresponds to a possible assignment of teachers to classes for one period. Our problem, therefore, is to partition the edges of G into as few matchings as possible or, equivalently, to properly colour the edges of G with as few colours as possible. Since G is bipartite, we know, by theorem 6.1, that $\chi'=\Delta$. Hence, if no teacher teaches for more than p periods, and if no class is taught for more than p periods, the teaching requirements can be scheduled in a p-period timetable. Furthermore, there is a good algorithm for constructing such a timetable, as is indicated in exercise 6.1.4. We thus have a complete solution to the timetabling problem.

However, the situation might not be so straightforward. Let us assume that only a limited number of classrooms are available. With this additional constraint, how many periods are now needed to schedule a complete timetable?

Suppose that altogether there are l lessons to be given, and that they have been scheduled in a p-period timetable. Since this timetable requires an average of l/p lessons to be given per period, it is clear that at least $\{l/p\}$ rooms will be needed in some one period. It turns out that one can always arrange l lessons in a p-period timetable so that at most $\{l/p\}$ rooms are occupied in any one period. This follows from theorem 6.3 below. We first have a lemma.

Lemma 6.3 Let M and N be disjoint matchings of G with $|M|>|N|$. Then there are disjoint matchings M' and N' of G such that $|M'|=|M|-1$, $|N'|=|N|+1$ and $M'\cup N'=M\cup N$.

Proof Consider the graph $H=G[M\cup N]$. As in the proof of theorem 5.1, each component of H is either an even cycle, with edges alternately in M and N, or else a path with edges alternately in M and N. Since $|M|>|N|$, some path component P of H must start and end with edges of M. Let $P=v_0e_1v_1\ldots e_{2n+1}v_{2n+1}$, and set

$$M'=(M\backslash\{e_1, e_3, \ldots, e_{2n+1}\})\cup\{e_2, e_4, \ldots, e_{2n}\}$$
$$N'=(N\backslash\{e_2, e_4, \ldots, e_{2n}\})\cup\{e_1, e_3, \ldots, e_{2n+1}\}$$

Then M' and N' are matchings of G that satisfy the conditions of the lemma $\square$

$$
\mathbf{P} = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \\ X_4 \end{array}
\begin{array}{c} \begin{array}{ccccc} Y_1 & Y_2 & Y_3 & Y_4 & Y_5 \end{array} \\
\left[\begin{array}{ccccc} 2 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \end{array}
$$

(a)

Period

	1	2	3	4
X_1	Y_1	Y_1	Y_3	Y_4
X_2	Y_2	–	Y_4	–
X_3	Y_3	Y_4	–	Y_2
X_4	Y_4	Y_5	–	–

(b)

Figure 6.4

Theorem 6.3 If G is bipartite, and if $p \geq \Delta$, then there exist p disjoint matchings $M_1, M_2, \ldots, M_p$ of G such that

$$E = M_1 \cup M_2 \cup \ldots \cup M_p \tag{6.4}$$

and, for $1 \leq i \leq p$

$$[\varepsilon/p] \leq |M_i| \leq \{\varepsilon/p\} \tag{6.5}$$

(Note: condition (6.5) says that any two matchings M_i and M_j differ in size by at most one.)

Proof Let G be a bipartite graph. By theorem 6.1, the edges of G can be partitioned into Δ matchings $M'_1, M'_2, \ldots, M'_\Delta$. Therefore, for any $p \geq \Delta$, there exist p disjoint matchings $M'_1, M'_2, \ldots, M'_p$ (with $M'_i = \emptyset$ for $i > \Delta$) such that

$$E = M'_1 \cup M'_2 \cup \ldots \cup M'_p$$

By repeatedly applying lemma 6.3 to pairs of these matchings that differ in size by more than one, we eventually obtain p disjoint matchings $M_1, M_2, \ldots, M_p$ of G satisfying (6.4) and (6.5), as required □

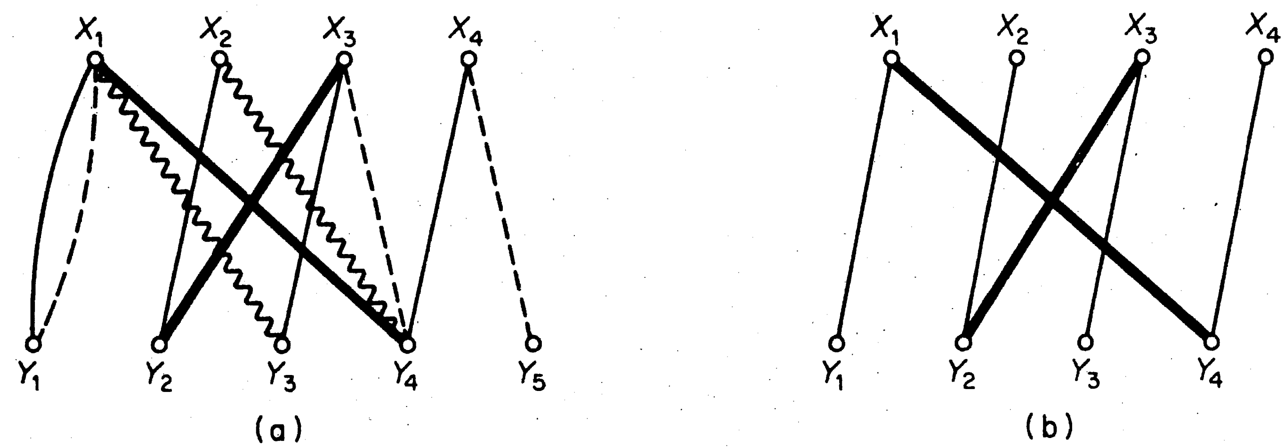
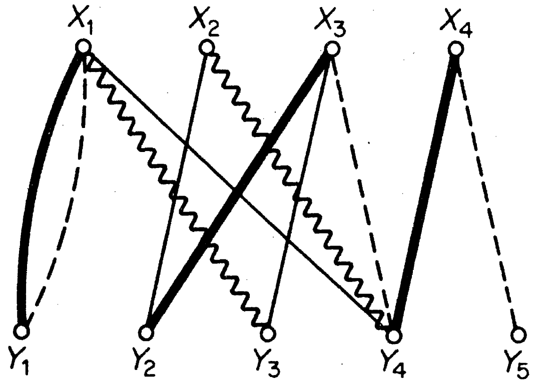


Figure 6.5



(a)

	Period 1	2	3	4
x_1	y_4	y_1	y_3	y_1
x_2	y_2	–	y_4	–
x_3	y_3	y_4	–	y_2
x_4	–	y_5	–	y_4

(b)

Figure 6.6

As an example, suppose that there are four teachers and five classes, and that the teaching requirement matrix $\mathbf{P} = [p_{ij}]$ is as given in figure 6.4*a*. One possible 4-period timetable is shown in figure 6.4*b*.

We can represent the above timetable by a decomposition into matchings of the edge set of the bipartite graph G corresponding to $\mathbf{P}$, as shown in figure 6.5*a*. (Normal edges correspond to period 1, broken edges to period 2, wavy edges to period 3, and heavy edges to period 4.)

From the timetable we see that four classes are taught in period 1, and so four rooms are needed. However $\varepsilon = 11$ and so, by theorem 6.4, a 4-period timetable can be arranged so that in each period either $2(=[11/4])$ or $3(=\{11/4\})$ classes are taught. Let M_1 denote the normal matching and M_4 the heavy matching; notice that $|M_1| = 4$ and $|M_4| = 2$. We can now find a 4-period 3-room timetable by considering $G[M_1 \cup M_4]$ (figure 6.5*b*). $G[M_1 \cup M_4]$ has two components, each consisting of a path of length three. Both paths start and end with normal edges and so, by interchanging the matchings on one of the two paths, we shall reduce the normal matching to one of three edges, and at the same time increase the heavy matching to one of three edges. If we choose the path $y_1x_1y_4x_4$, making the edges y_1x_1 and y_4x_4 heavy and the edge x_1y_4 normal, we obtain the decomposition of E shown in figure 6.6*a*. This then gives the revised timetable shown in figure 6.6*b*; here, only three rooms are needed at any one time.

	Period 1	2	3	4	5	6
x_1	y_4	y_3	y_1	–	y_1	–
x_2	y_2	y_4	–	–	–	–
x_3	–	–	y_4	y_3	y_2	–
x_4	–	–	–	y_4	–	y_5

Figure 6.7

However, suppose that there are just two rooms available. Theorem 6.4 tells us that there must be a 6-period timetable that satisfies our requirements (since $\{11/6\}=2$). Such a timetable is given in figure 6.7.

In practice, most problems on timetabling are complicated by preassignments (that is, conditions specifying the periods during which certain teachers and classes must meet). This generalisation of the timetabling problem has been studied by Dempster (1971) and de Werra (1970).

Exercise

6.3.1 In a school there are seven teachers and twelve classes. The teaching requirements for a five-day week are given by the matrix

$$
\begin{array}{r|cccccccccccc}
 & Y_1 & Y_2 & Y_3 & Y_4 & Y_5 & Y_6 & Y_7 & Y_8 & Y_9 & Y_{10} & Y_{11} & Y_{12} \\
\hline
X_1 & 3 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
X_2 & 1 & 3 & 6 & 0 & 4 & 2 & 5 & 1 & 3 & 3 & 0 & 4 \\
X_3 & 5 & 0 & 5 & 5 & 0 & 0 & 5 & 0 & 5 & 0 & 5 & 5 \\
\mathbf{P} = X_4 & 2 & 4 & 2 & 4 & 2 & 4 & 2 & 4 & 2 & 4 & 2 & 3 \\
X_5 & 3 & 5 & 2 & 2 & 0 & 3 & 1 & 4 & 4 & 3 & 2 & 5 \\
X_6 & 5 & 5 & 0 & 0 & 5 & 5 & 0 & 5 & 0 & 5 & 5 & 0 \\
X_7 & 0 & 3 & 4 & 3 & 4 & 3 & 4 & 3 & 4 & 3 & 3 & 0
\end{array}
$$

where p_{ij} is the number of periods that teacher X_i must teach class Y_j..

(*a*) Into how many periods must a day be divided so that the requirements can be satisfied?

(*b*) If an eight-period/day timetable is drawn up, how many classrooms will be needed?

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