CSCI 1900
Discrete Structures

Logical Operations
Reading: Kolman, Section 2.1

Statement of Proposition

- Statement of proposition – a declarative sentence that is either true or false, but not both
- Examples:
  - The earth is round: statement that is true
    2+3=5: statement that is true
  - Do you speak English? This is a question, not a statement

More Examples of Statements of Proposition

- 3-x=5: is a declarative sentence, but not a statement since it is true or false depending on the value of x
- Take two aspirins: is a command, not a statement
- The temperature on the surface of the planet Venus is 800^{o}F: is a declarative statement of whose truth is unknown to us
- The sun will come out tomorrow: a statement that is either true or false, but not both, although we will have to wait until tomorrow to determine the answer

Logical Connectives and Compound Statements

- *x, y, z, …* denote variables that can represent real numbers
- *p, q, r,…* denote prepositional variables that can be replaced by statements.
  - *p*: The sun is shining today
  - *q*: It is cold

Negation

- If *p* is a statement, the negation of *p* is the statement *not p*
- Denoted ~*p*
- If *p* is true, ~*p* is false
- If *p* is false, ~*p* is true
- ~*p* is not actually connective, i.e., it doesn't join two of anything
- ***not*** is a unary operation for the collection of statements and ~*p* is a statement if *p* is

Examples of Negation

- If *p*: 2+3 >1 then If ~*p*: 2+3 $\leq$1
- If *q*: It is cold then ~*q*: It is not the case that it is cold, i.e., It is not cold.

Conjunction

- If p and q are statements, then the ***conjunction*** of p and q is the compound statement “p and q”
- Denoted $p \wedge q$
- $p \wedge q$ is true only if both p and q are true
- Example:
  - p: ETSU parking permits are expensive
  - q: ETSU has plenty of parking
  - $p \wedge q$ = ?

Disjunction

- If p and q are statements, then the ***disjunction*** of p and q is the compound statement “p or q”
- Denoted $p \vee q$
- $p \vee q$ is true if either p or q are true
- Example:
  - p: I am a male
  - q: I am under 40 years old
  - $p \vee q$ = ?

Exclusive Disjunction

- If p and q are statements, then the ***exclusive disjunction*** is the compound statement, “either p or q may be true, but both are not true at the same time.”
- Example:
  - p: It is daytime
  - q: It is night time
  - $p \vee q$ (in the exclusive sense) = ?

Inclusive Disjunction

- If p and q are statements, then the ***inclusive disjunction*** is the compound statement, “either p or q may be true or they may both be true at the same time.”
- Example:
  - p: It is cold
  - q: It is night time
  - $p \vee q$ (in the inclusive sense) = ?

Exclusive versus Inclusive

- Depending on the circumstances, some disjunctions are inclusive and some of exclusive.
- Examples of Inclusive
  - “I have a dog” or “I have a cat”
  - “It is warm outside” or “It is raining”
- Examples of Exclusive
  - Today is either Tuesday or it is Thursday
  - Pat is either male or female

Compound Statements

- A ***compound statement*** is a statement made from other statements
- For n individual propositions, there are 2^n possible combinations of truth values
- A truth table contains 2^n rows identifying the truth values for the statement represented by the table.
- Use parenthesis to denote order of precedence
- $\wedge$ has precedence over $\vee$

Truth Tables are Important Tools for this Material!

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Compound Statement Example $(p \wedge q) \vee (\sim p)$

p	q	$p \wedge q$	$\sim p$	$(p \wedge q) \vee (\sim p)$
T	T	T	F	T
T	F	F	F	F
F	T	F	T	T
F	F	F	T	T

Quantifiers

- Back in Section 1.1, a set was defined $\{x \mid P(x)\}$
- For an element t to be a member of the set, P(t) must evaluate to "true"
- P(x) is called a predicate or a propositional function

Computer Science Functions

- if P(x), then execute certain steps
- while Q(x), do specified actions

Universal quantification of a predicate P(x)

- Universal quantification of predicate P(x) = For all values of x, P(x) is true
- Denoted $\forall x\, P(x)$
- The symbol $\forall$ is called the universal quantifier
- The order in which multiple quantifications are considered does not affect the truth value (e.g., $\forall x\, \forall y\, P(x,y) \equiv \forall y\, \forall x\, P(x,y)$)

Examples:

- P(x): $-(-x) = x$
  - This predicate makes sense for all real numbers x.
  - The universal quantification of P(x), $\forall x\, P(x)$, is a true statement, because for all real numbers, $-(-x) = x$
- Q(x): $x+1<4$
  - $\forall x\, Q(x)$ is a false statement, because, for example, Q(5) is not true

Existential quantification of a predicate P(x)

- Existential quantification of a predicate P(x) is the statement "There exists a value of x for which P(x) is true."
- Denoted ∃x P(x)
- Existential quantification may be applied to several variables in a predicate
- The order in which multiple quantifications are considered does not affect the truth value

Applying both universal and existential quantification

- Order of application does matter
- Example: Let ***A*** and ***B*** be n x n matrices
- The statement $\forall \boldsymbol{A}\ \exists \boldsymbol{B}\ \ \boldsymbol{A} + \boldsymbol{B} = \boldsymbol{I}_n$
- Reads "for every ***A*** there is a ***B*** such that $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{I}_n$"
- Prove by coming up for equations for b_{ii} and b_{ij} ($j \neq i$)
- Now reverse the order: $\exists \boldsymbol{B}\ \forall \boldsymbol{A}\ \boldsymbol{A} + \boldsymbol{B} = \boldsymbol{I}_n$
- Reads "there exists a ***B*** such that for all ***A*** $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{I}_n$"
- THIS IS FALSE!

Assigning Quantification to Proposition

- Let p: ∀x P(x)
- The negation of p is false when p is true and true when p is false
- For p to be false, there must be at least one value of x for which P(x) is false.
- Thus, p is false if ∃x ~P(x) is true.
- If ∃x ~P(x) is false, then for every x, ~P(x) is false; that is ∀x P(x) is true.

Okay, what exactly did the previous slide say?

- Assume a statement is made that "for all x, P(x) is true."
  - If we can find one case that is not true, then the statement is false.
  - If we cannot find one case that is not true, then the statement is true.
- Example: ∀ positive integers, n, $P(n) = n^2 + 41n + 41$ is a prime number.
  - This is false because ∃ an integer resulting in a non-prime value, i.e., ∃n such that P(n) is false.