CSCI 1900
Discrete Structures

Permutations

Reading: Kolman, Section 3.1

Sequences Derived from a Set

- Assume we have a set A containing n items.
- Examples include alphabet, decimal digits, playing cards, etc.
- We can produce sequences from each of these sets.

Types of Sequences from a Set

There are a number of different ways to create a sequence from a set

- Any order, duplicates allowed
- Any order, no duplicates allowed
- Order matters, duplicates allowed
- Order matters, no duplicates allowed

Classifying Real-World Sequences

Determine size of set A and classify each of the following as one of the previously listed types of sequences

- Five card stud poker
- Phone numbers
- License plates
- Lotto numbers
- Binary numbers
- Windows XP CD Key
- Votes in a presidential election
- Codes for 5-digit CSCI door locks

Multiplication Principle of Counting

- The first type of sequence we will look at is where duplicates are allowed and their order matters.
- Supposed that two tasks T_1 and T_2 must be performed in sequence.
- If T_1 can be performed in n_1 ways, and for each of these ways, T_2 can be performed in n_2 ways, then the sequence T_1T_2 can be performed in $n_1 \cdot n_2$ ways.

Multiplication Principle (continued)

- Extended previous example to T_1, T_2, …, T_k
- Solution becomes $n_1 \cdot n_2 \cdot \ldots \cdot n_k$

Examples of Multiplication Principle

- 8 character passwords
  - First digit must be a letter
  - Any character after that can be a letter or a number
  - 26*36*36*36*36*36*36*36 = 2,037,468,266,496
- Windows XP/2000 software keys
  - 25 characters of letters or numbers
  - 36^{25}

More Examples of Multiplication Principle

- License plates of the form "ABC 123":
  - 26*26*26*10*10*10 = 17,576,000
- Phone numbers
  - Three digit area code cannot begin with 0
  - Three digit exchange cannot begin with 0
  - 9*10*10*9*10*10*10*10*10*10 = 8,100,000,000

Calculation of the Number of Subsets

- Let A be a set with n elements: how many subsets does A have?
- Each element may either be included or not included.
- In section 1.3, we talked about the characteristic function which defines membership in a set based on a universal set
- Example:
  - U={1,2,3,4,5,6}
  - A = {1,2}, B={2, 4, 6}
  - f_A = {1,1,0,0,0,0}, f_B = {0,1,0,1,0,1}

Calculation of the Number of Subsets (continued)

- Every subset of A can be defined with a characteristic function of n elements where each element is a 1 or a 0, i.e., each element has 2 possible values
- Therefore, there are $2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$ possible characteristic functions

Permutations

- The next type of sequence we will look at is where duplicates are not allowed and their order matters
- Assume A is a set of n elements
- Suppose we want to make a sequence, S, of length r where $1 \leq r \leq n$

Multiplication Principle Versus Permutations

- If repeated elements are allowed, how many different sequences can we make?
- Process:
  - Each time we select an element for the next element in the sequence, S, we have n to choose from
  - This gives us $n \cdot n \cdot n \cdot \ldots \cdot n = n^r$ possible choices

Multiplication Principle Versus Permutations (continued)

- Suppose repeated elements are ***not*** allowed, how many different sequences can we make?
- Process:
  - The first selection, T_1, provides n choices.
  - Each time we select an element after that, T_k where k>1, there is one less than there was for the previous selection, k-1.
  - The last choice, T_r, has n – (r – 1) = n – r + 1 choices
  - This gives us **n·(n – 1)·(n – 2)·…·(n – r + 1)**

Permutations

- Notation: ${}_nP_r$ is called number of permutations of n objects taken r at a time.
- Word scramble: How many 4 letter words can be made from the letters in "Gilbreath" without duplicate letters?

$${}_9P_4 = 9·8·7·6 = 3{,}024$$

- Example, how many 4-digit PINs can be created for the 5 button CSCI door locks?

$${}_5P_4 = 5·4·3·2 = 120$$

- Would adding a fifth digit give us more PINs?

Factorial

- For r=n,
  ${}_nP_n = {}_nP_r$ = n·(n – 1)·(n – 2)·…·2·1
- This number is also written as ***n!*** and is read ***n factorial***
- ${}_nP_r$ can be written in terms of factorials

$${}_nP_r = n·(n – 1)·(n – 2)·…·(n – r + 1)$$

$${}_nP_r = \frac{n·(n – 1)·(n – 2)·…·(n – r + 1)·(n – r)·…·2·1}{(n – r)·…·2·1}$$

$${}_nP_r = n!/(n – r)!$$

Distinguishable Permutations from a Set with Repeated Elements

- If the set from which a sequence is being derived has duplicate elements, e.g., {a, b, d, d, g, h, r, r, r, s, t}, then straight permutations will actually count some sequences multiple times.
- Example: How many words can be made from the letters in Tarnoff?
- Problem: the f's cannot be distinguished, e.g., aorf cannot be distinguished from aorf

Distinguishable Permutations from a Set with Repeated Elements

- Number of distinguishable permutations that can be formed from a collection of n objects where the first object appears k_1 times, the second object k_2 times, and so on is:

$$n! / (k_1!· k_2!··· k_t!)$$

where $k_1 + k_2 + … + k_t = n$

Example

- How many distinguishable words can be formed from the letters of JEFF?
- Solution: n = 4, $k_j = 1$, $k_e = 1$, $k_f = 2$

$$n!/(k_j! · k_e! · k_f!) = 4!/(1!\ 1!\ 2!) = 12$$

- List:
  JEFF, JFEF, JFFE, EJFF, EFJF, EFFJ, FJEF, FEJF, FJFE, FEFJ, FFJE, and FFEJ

Example

- How many distinguishable words can be formed from the letters of MISSISSIPPI?
- Solution:
  $n = 11, k_m = 1, k_i = 4, k_s = 4, k_p = 2$

  $n!/(k_m! \cdot k_i! \cdot k_s! \cdot k_p!) = 11!/(1!\ 4!\ 4!\ 2!)$
  $= 34,650$

In-Class Exercises

- How many ways can you sort a deck of 52 cards?
- Compute the number of 4-digit ATM PINs where duplicate digits are allowed.
- How many ways can the letters in the word "TARNOFF" be arranged?