Appendix IV
Unsolved Problems

Collected here are a number of unsolved problems of varying difficulty, with origins, dates and relevant bibliography. Conjectures are displayed in bold type. Problems marked † have now been solved; see page 253.

1. Two graphs $G$ and $H$ are hypomorphic (written $G \equiv H$) if there is a bijection $\sigma: V(G) \rightarrow V(H)$ such that $G - v \equiv H - \sigma(v)$ for all $v \in V(G)$. A graph $G$ is reconstructible if $G \equiv H$ implies $G \equiv H$. The reconstruction conjecture claims that every graph $G$ with $\nu > 2$ is reconstructible (S. M. Ulam, 1929). This has been verified for disconnected graphs, trees and a few other classes of graphs (see Harary, 1974).

There is a corresponding edge reconstruction conjecture: every graph $G$ with $\epsilon > 3$ is edge reconstructible. Lovász (1972) has shown that every simple graph $G$ with $\epsilon > \binom{\nu}{2}/2$ is edge reconstructible.

P. K. Stockmeyer has found an infinite family of counterexamples to the analogous reconstruction conjecture for digraphs.


2. A graph $G$ is embeddable in a graph $H$ if $G$ is isomorphic to a subgraph of $H$. Characterise the graphs embeddable in the $k$-cube (V. V. Firsov, 1965).


3. Every 4-regular simple graph contains a 3-regular subgraph (N. Sauer, 1973).

4. If $k > 2$, there exists no graph with the property that every pair of vertices is connected by a unique path of length $k$ (A. Kotzig, 1974). Kotzig has verified his conjecture for $k < 9$.

5. Every connected graph $G$ is the union of at most $[(\nu+1)/2]$ edge-disjoint paths (T. Gallai, 1962). Lovász (1968) has shown that every graph $G$ is the union of at most $[\nu/2]$ edge-disjoint paths and cycles.
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7. If $G$ is a simple block with at least $v/2 + k$ vertices of degree at least $k$, then $G$ has a cycle of length at least $2k$ (D. R. Woodall, 1975).

8. Let $f(m, n)$ be the maximum possible number of edges in a simple graph on $n$ vertices which contains no $m$-cycle. It is known that

$$f(m, n) = \begin{cases} 
\left\lfloor \frac{n^2}{4} \right\rfloor & \text{if } m \text{ is odd, } m \leq \frac{1}{2}(n + 3) \\
\left( \frac{n - m + 2}{2} \right) + \binom{m - 1}{2} & \text{if } m \geq \frac{1}{2}(n + 3)
\end{cases}$$

Determine $f(m, n)$ for the remaining cases (P. Erdős, 1963).


9. Let $f(n)$ be the maximum possible number of edges in a simple graph on $n$ vertices which contains no 3-regular subgraph. Determine $f(n)$ (P. Erdős and N. Sauer, 1974). Since there is a constant $c$ such that every simple graph $G$ with $\varepsilon \geq cn^{8/5}$ contains the 3-cube (Erdős and Simonovits, 1970), clearly $f(n) < cn^{8/5}$.


10. Determine which simple graphs $G$ have exactly one cycle of each length $l$, $3 \leq l \leq v$ (R. C. Entringer, 1973).

11. Let $f(n)$ be the maximum possible number of edges in a graph on $n$ vertices in which no two cycles have the same length. Determine $f(n)$ (P. Erdős, 1975).

12. If $G$ is simple and $\varepsilon > v(k - 1)/2$, then $G$ contains every tree with $k$ edges (P. Erdős and V. T. Sós, 1963). It is known that every such graph contains a path of length $k$ (Erdős and Gallai, 1959).


13. Find a $(6, 5)$-cage (see appendix III).
14. The bandwidth of $G$ is defined to be

$$\min_{l} \max_{u,v \in E} |l(u) - l(v)|$$

where the minimum is taken over all labellings $l$ of $V$ in distinct integers. Find bounds for the bandwidth of a graph (L. H. Harper, 1964). The bandwidth of the $k$-cube has been determined by Harper (1966).


15. A simple graph $G$ is *graceful* if there is a labelling $l$ of its vertices with distinct integers from the set $\{0, 1, \ldots, \varepsilon\}$, so that the induced edge labelling $l'$ defined by

$$l'(uv) = |l(u) - l(v)|$$

assigns each edge a different label. Characterise the graceful graphs (S. Golomb, 1972). It has been conjectured that, in particular, every tree is *graceful* (A. Rosa, 1966).


16. The 3-connected planar graph on $2m$ edges with the least possible number of spanning trees is the wheel with $m$ spokes (W. T. Tutte, 1940).


17. Let $u$ and $v$ be two vertices in a graph $G$. Denote the minimum number of vertices whose deletion destroys all $(u, v)$-paths of length at most $n$ by $a_n$, and the maximum number of internally disjoint $(u, v)$-paths of length at most $n$ by $b_n$. Let $f(n)$ denote the maximum possible value of $a_n/b_n$. Determine $f(n)$ (V. Neumann, 1974). L. Lovász has conjectured that $f(n) \leq \sqrt{n}$. It is known that

$$\lceil \sqrt{n/2} \rceil \leq f(n) \leq \lfloor n/2 \rfloor$$

18. Every 3-regular 3-connected bipartite planar graph is *hamiltonian* (D. Barnette, 1970). P. Goodey has verified this conjecture for plane graphs whose faces are all of degree four or six. Note that if the planarity condition is dropped, the conjecture is no longer valid (see appendix III).

19. A graphic sequence $\mathbf{d}$ is *forcibly hamiltonian* if every simple graph with degree sequence $\mathbf{d}$ is hamiltonian. Characterise the forcibly hamiltonian
sequences (C. St. J. A. Nash-Williams, 1970). (Theorem 4.5 gives a partial solution.)


20. Every connected vertex-transitive graph has a Hamilton path (L. Lovász, 1968). L. Babai has verified this conjecture for graphs with a prime number of vertices.

21. A graph $G$ is $t$-tough if, for every vertex cut $S$, $\omega(G - S) \leq |S|/t$. (Thus theorem 4.2 says that every Hamiltonian graph is 1-tough.)

(a) If $G$ is 2-tough, then $G$ is hamiltonian (V. Chvátal, 1971). C. Thomassen has obtained an example of a nonhamiltonian $t$-tough graph with $t > 3/2$.

(b) If $G$ is 3/2-tough, then $G$ has a 2-factor (V. Chvátal, 1971).


22. The binding number of $G$ is defined by

$$\text{bind } G = \min_{g \in G} \left| \frac{|N(S)|}{|S|} \right|$$

(a) If $\text{bind } G \geq 3/2$, then $G$ contains a triangle (D. R. Woodall, 1973).

(b) If $\text{bind } G \geq 3/2$, then $G$ is pancyclic (contains cycles of all lengths $l$, $3 \leq l \leq n$) (D. R. Woodall, 1973).

Woodall (1973) has shown that $G$ is hamiltonian if $\text{bind } G \geq 3/2$, and that $G$ contains a triangle if $\text{bind } G \geq 1/2(1 + \sqrt{5})$.


23. Every nonempty regular simple graph contains two disjoint maximal independent sets (C. Payan, 1973)

24. Find the Ramsey number $r(3, 3, 3, 3)$. It is known that

$$51 \leq r(3, 3, 3, 3) \leq 65$$


25. For $m < n$, let $f(m, n)$ denote the least possible number of vertices in a graph which contains no $K_n$ but has the property that in every 2-edge colouring there is a monochromatic $K_m$. (Folkman, 1970 has established the existence of such graphs.) Determine bounds for $f(m, n)$. It is
known that
\[ f(3, n) = 6 \quad \text{for} \quad n \geq 7 \]
\[ f(3, 6) = 8 \quad (\text{see exercise 7.2.5}) \]
\[ 10 \leq f(3, 5) \leq 18 \]

Folkman, J. (1970). Graphs with monochromatic complete subgraphs in
graph-colouring constant. *J. Combinatorial Theory* (B), 15, 200–203
Lin, S. On Ramsey numbers and \( K_r \)-coloring of graphs. *J. Combinatorial
Theory* (B), 12, 82–92

26. **If** \( G \) **is** \( n \)-**chromatic, then** \( r(G, G) \geq r(n, n) \) (P. Erdős, 1973). \( (r(G, G) \) is
defined in exercise 7.2.6.)

27. What is the maximum possible chromatic number of a graph which can
be drawn in the plane so that each edge is a straight line segment of unit

graph. *Mathematika*, 12, 118–22

28. **The absolute values of the coefficients of any chromatic polynomial form
a unimodal sequence** (that is, no term is flanked by terms of greater

Combinatorial Theory*, 9, 95–96

29. **If** \( G \) **is not complete and** \( \chi = m + n - 1 \), where \( m \geq 2 \) and \( n \geq 2 \), then
**there exist disjoint subgraphs** \( G_1 \) **and** \( G_2 \) **of** \( G \) **such that** \( \chi(G_1) = m \) **and
**\( \chi(G_2) = n \) (L. Lovász, 1968).

30. A simple graph \( G \) is **perfect** if, for every induced subgraph \( H \) of \( G \), the
number of vertices in a maximum clique is \( \chi(H) \). **\( G \) is perfect if and only if no induced subgraph of** \( G \) **or** \( G^c \) **is an odd cycle of length greater
than three** (C. Berge, 1961). This is the **strong perfect graph conjecture**.
Lovász (1972) has shown that the complement of any perfect graph is
perfect.

Parthasarathy, K. R. and Ravindra, G. (to be published). The strong
perfect-graph conjecture is true for \( K_{1,3} \)-free graphs. *J. Combinatorial Theory*

31. **If** \( G \) **is a 3-regular simple block and** \( H \) **is obtained from** \( G \) **by
duplicating each edge, then** \( \chi'(H) = 6 \) (D. R. Fulkerson, 1971).

32. **If** \( G \) **is simple, with** \( \nu \) **even and** \( \chi(G) = \Delta(G) + 1 \), then** \( \chi'(G - \nu) = \chi'(G) \)
for some $v \in V$ (I. T. Jakobsen, L. W. Beineke and R. J. Wilson, 1973). This has been verified for all graphs $G$ with $v \leq 10$ and all 3-regular graphs $G$ with $v = 12$.


33. **For any simple graph $G$, the elements of $V \cup E$ can be coloured in $\Delta + 2$ colours so that no two adjacent or incident elements receive the same colour** (M. Behzad, 1965). This is known as the total colouring conjecture. M. Rosenfeld and N. Vijayaditya have verified it for all graphs $G$ with $\Delta \leq 3$.


34. **If $G$ is simple and $\varepsilon > 3v - 6$, then $G$ contains a subdivision of $K_5$** (G. A. Dirac, 1964). Thomassen (1975) has shown that $G$ contains a subdivision of $K_5$ if $\varepsilon \geq 4v - 10$.


35. A sequence $\mathbf{d}$ of non-negative integers is potentially planar if there is a simple planar graph with degree sequence $\mathbf{d}$. Characterise the potentially planar sequences (S. L. Hakimi, 1963).


†36. **If $G$ is a loopless planar graph, then $\alpha \geq v/4$** (P. Erdős, 1968). Albertson (1974) has shown that every such graph satisfies $\alpha > 2v/9$.


†37. **Every planar graph is 4-colourable** (F. Guthrie, 1852).


38. **Every $k$-chromatic graph contains a subgraph contractible to $K_k$** (H. Hadwiger, 1943). Dirac (1964) has proved that every 6-chromatic graph contains a subgraph contractible to $K_6$ less one edge.


39. **Every $k$-chromatic graph contains a subdivision of $K_k$** (G. Hajós, 1961). Pelikán (1969) has shown that every 5-chromatic graph contains a subdivision of $K_5$ less one edge.

40. Every 2-edge-connected 3-regular simple graph which has no Tait colouring contains a subgraph contractible to the Petersen graph (W. T. Tutte, 1966).


41. For every surface S, there exists a finite number of graphs which have minimum degree at least three and are minimally nonembeddable on S.

†42. If D is disconnected, then D has a directed cycle of length at least χ (M. Las Vergnas, 1974).

43. If D is a tournament with ν odd and every indegree and outdegree equal to (ν−1)/2, then D is the union of (ν−1)/2 arc-disjoint directed Hamilton cycles (P. Kelly, 1966).

44. If D is a tournament with ν even, then D is the union of ∑ν max{0, d*(ν) − d−(ν)} arc-disjoint directed paths (R. O’Brien, 1974).

This would imply the truth of conjecture 43.

45. Characterise the tournaments D with the property that all subtournaments D − ν are isomorphic (A. Kotzig, 1973).

46. If D is a digraph which contains a directed cycle, then there is some arc whose reversal decreases the number of directed cycles in D (A. Adám, 1963).

47. Given a positive integer n, there exists a least integer f(n) such that in any digraph with at most n arc-disjoint directed cycles there are f(n) arcs whose deletion destroys all directed cycles (T. Gallai, 1964; D. H. Younger, 1968).


48. An (m + n)-regular graph is (m, n)-orientable if it can be oriented so that each indegree is either m or n. Every 5-regular simple graph with no 1-edge cut or 3-edge cut is (4, 1)-orientable (W. T. Tutte, 1972). Tutte has shown that this would imply Grötzsch’s theorem

49. Obtain an algorithm to find a maximum flow in a network with two sources x₁ and x₂, two sinks y₁ and y₂, and two commodities, the requirement being to ship commodity 1 from x₁ to y₁ and commodity 2 from x₂ to y₂ (L. R. Ford and D. R. Fulkerson, 1962).

50. **Every 2-edge-connected digraph** D **has a circulation** f **over the field of integers modulo 5 in which** f(α) ≠ 0 **for all arcs** α **(W. T. Tutte, 1949).** Tutte has shown that this would imply the five-colour theorem.

References for problems solved since first printing:

