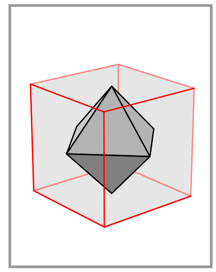
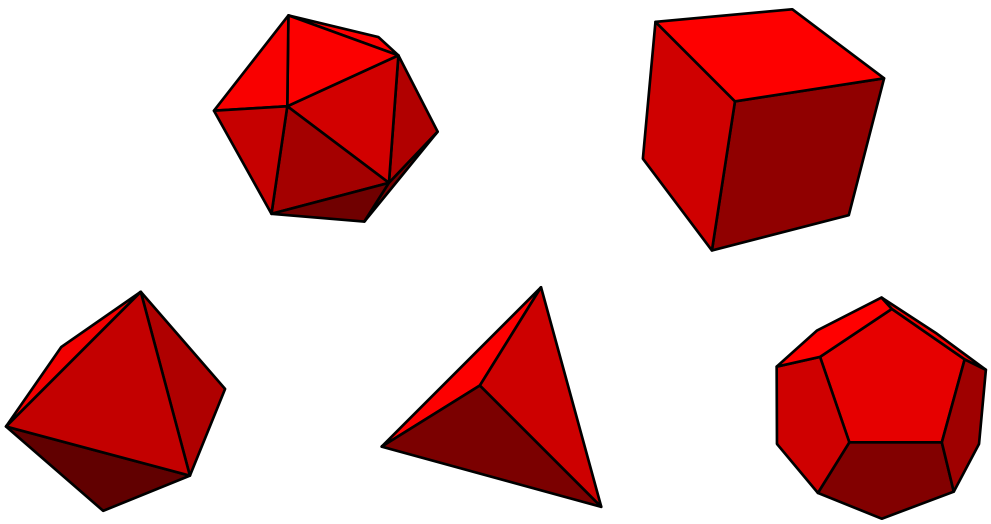


CHAPTER 15

The regular polyhedra

A regular figure is one which is . . . well, more *regular* than most. A polyhedron is a shape in three dimensions olyhedron:regular:1 whose surface is a collection of flat polygons, and **a regular polyhedron** is one all of whose faces and vertices look the same. It has been known for a very long time that there are exactly five regular polyhedra. Although they are favourites for computer graphics, they are probably not clearly understood by those who draw them.



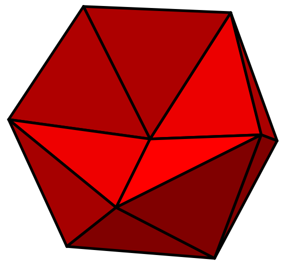
That there are no more than five is by no means a trivial fact, although it is one to which we have become accustomed. The regular polyhedra have been known for a very long time as mathematical history goes—the oldest extant Greek mathematical text, some scribbling on discarded pottery discovered on Elephantine Island near the Aswan cataracts that dates to about 250 B.C., is concerned with them. Their properties are in fact not easy to understand, and perhaps familiarity has made it more difficult to realize how remarkable they are. Although it is perhaps not the most mathematically sophisticated part of Euclid, the regular polyhedra are discussed only Elements:Euclid's:1 in the last book of the *Elements*, and the treatment is not at all transparent. In order to show how an extended graphical reconstruction of Euclid can go, I will sketch his treatment in this Chapter.

There are two quite different parts of the story: (1) It is possible to construct five different regular polyhedra; (2) it is not possible to construct any others. Exactly what these assertions mean, and how very distinct they are, will be appreciated later on. For both, at least to start with, I shall follow Euclid rather closely. At an elementary level, it is a hard act to beat.

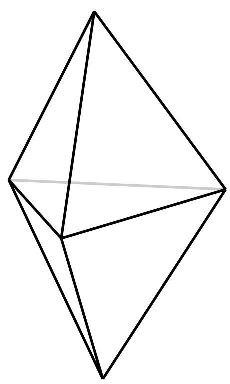
I shall begin with part (2), and deal with the construction later on. But first a few more opening remarks.

1. What exactly is a regular polyhedron?

It is important first to understand exactly what a regular polyhedron is. The first, and simplest condition, is that its faces are to be regular polygons. Another is that all of these faces be congruent to each other. But there has to be some extra condition to guarantee regularity. For example, these two conditions of facial congruence will be satisfied by an icosahedron with some of its sides pushed in, which surely wouldn't be considered to be all that regular:



So there has to be something more required. For one thing, the figure should be **convex**, which means loosely that it bulges out. Technically this means that any two points in the figure can be joined by a segment contained completely inside the figure itself. At any rate, this new condition certainly excludes the mutant above. But it isn't sufficient to characterize regularity either, since the following figure, which is constructed by gluing two regular tetrahedra together, shouldn't qualify as regular.



This suggests that we impose the condition that all the vertices of the figure, as well as all its faces, 'look alike' in the sense that they are congruent. We shall in all specify:

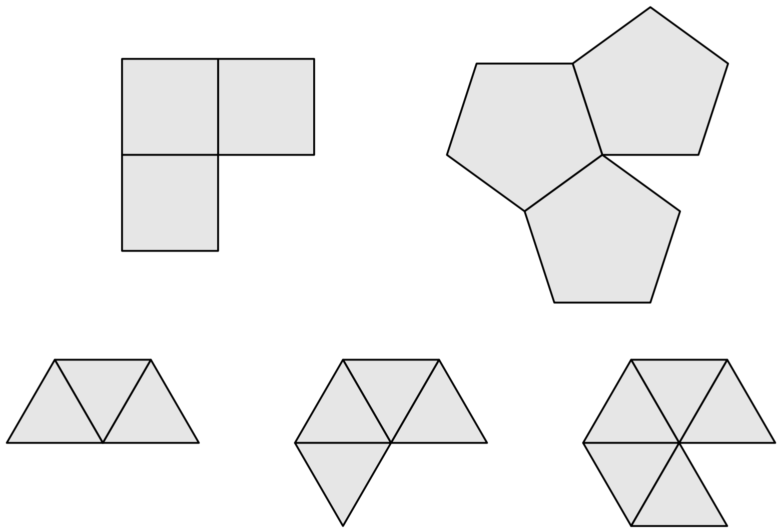
- *By definition, a regular polyhedron is one satisfying all four of these conditions:*
  - *(a) All of its faces are regular polygons;*
  - *(b) they are all congruent to one another;*
  - *(c) the figure is convex;*
  - *(d) all of its vertices are congruent.*

In fact, these conditions are unnecessarily strong. It is actually the case that we need only require that the number of faces around each vertex be the same for all vertices. A remarkable theorem proven by the French **Cauchy:A.:2** mathematician Cauchy in the early nineteenth century asserts that these conditions are redundant. But it is not a simple result, and it is better in an elementary treatment not to depend on it.

2. There are no more than five regular solids

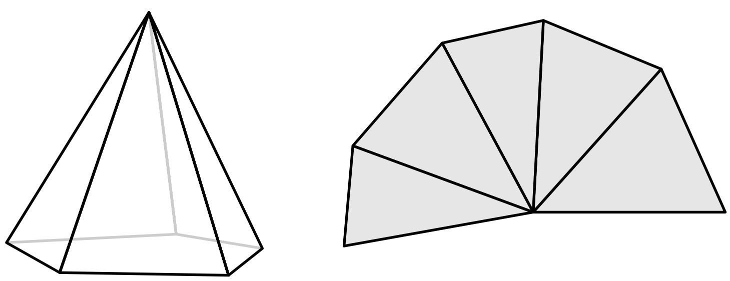
Euclid's, Book XIII:2 I shall first explain roughly why there cannot be more than five regular polyhedra, and then go over the argument later in detail. Incidentally, this assertion is somewhat informally inserted at the end of Book XIII of Euclid's *Elements*, even though many of his earlier results are clearly leading up to it.

That there cannot be more than five regular solids just depends essentially on what happens around one of the vertices, call it P, of a regular polyhedron. Throw away all of the faces of the polyhedron which do not touch P. Then flatten out the faces that are left. For the polyhedra we know about, we get the following pictures of what I call the **splayed** vertices.



Proposition XI.21:3 It is intuitively reasonably clear that when we do that, the vertex 'opens up' in the sense that in going around the vertex we don't go all around its image in the plane. In fact, this is a special case of Proposition XI.21 from Euclid, which is much more general:

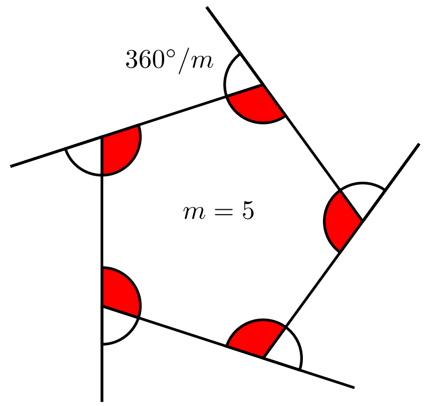
- (Euclid XI.21) *In going once around the faces touching any convex vertex, the angles traversed always add up to less than* $360°$.



The word 'convex' here, as with the earlier use of the same word, means a vertex which always bulges out. Convexity is clearly a necessary condition, since if we are allowed to fold up the faces around a vertex like an accordion the proposition is no longer true. I shall come back later to give the details of the argument, nearly all of which arise in the very beginning (Book I) of Euclid. Let's assume for now that the result is true and see why it implies that there can be no more than five regular polyhedra.

Since all faces of a regular polyhedron are congruent, each one of them will be a regular polygon with some fixed number of sides. Furthermore, all the vertices are congruent, which means that the number of faces touching each vertex must be the same. Suppose that each face has m sides, and that each vertex is touched by n faces. What are the possibilities?

- *In a regular plane polygon of m sides, the angle at each corner is equal to $180° - 360°/m$.*

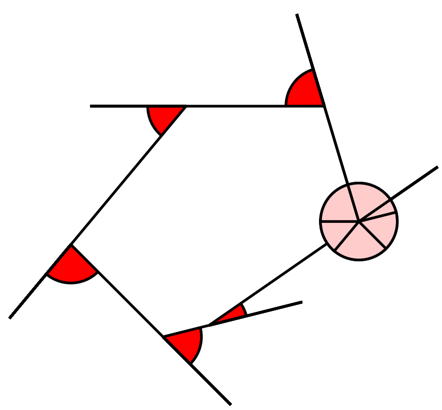


Since the inside angle at any corner is $180°$ less the angle turned at that corner, this follows immediately from a more intuitive result:

s, Proposition I.32:4

- (Euclid I.32) *If we follow around the outside of a convex plane polygon, the total angle turned is* $360°$.

This is intuitively clear, but it can be reduced to something really apparent by translating these angles to one vertex:



At any rate, if we have n polygons at a single convex vertex, each of m sides, then the total angle traversed as we go around the faces next to the vertex will be $n\,(180° - 360°/m)$ and this must be less than 360:

$$n(1 - 2/m)\,180° < 360°$$
$$180° - \frac{360°}{m} < \frac{360°}{n}$$

which leads to the inequality

$$\frac{1}{2} < \frac{1}{m} + \frac{1}{n}\,.$$

Each face must be at least a triangle, so $m \geq 3$. The number of faces meeting a true vertex has to be at least three, hence $n \geq 3$ as well. Therefore

$$\begin{aligned} \frac{1}{2} &< \frac{1}{m} + \frac{1}{n} \\ &\leq \frac{1}{m} + \frac{1}{3} \\ \frac{1}{6} &< \frac{1}{m} \\ n &< 6\,. \end{aligned}$$

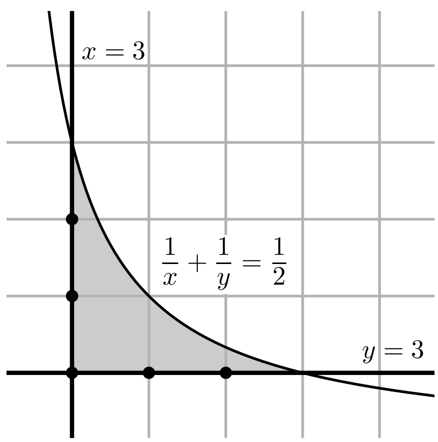
Similarly, $m < 6$. So we have only a finite number of possibilities to look at, examined in the following table, which shows $1/m + 1/n$ for $3 \leq m, n < 6$, except that those that don't qualify are left out:

Values of $1/m + 1/n > 1/2$

m \ n	3	4	5	6
3	$2/3$	$7/12$	$8/15$	–
4	$7/12$	–	–	–
5	$8/15$	–	–	–
6	–	–	–	–

We see that there are exactly five possibilities, each corresponding to one of the known regular polyhedra.

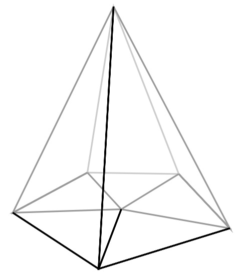
Another way to see which m and n qualify is to sketch the region $1/x+1/y > 1/2$, $x \geq 3$, $y \geq 3$ in the (x, y)-plane, and observe which points with integral coordinates lie inside it.



3. The proof of Euclid XI.21

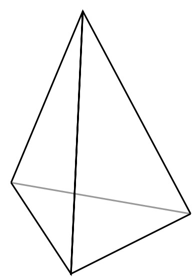
Proposition XI.21:5 The proof that Euclid gives for Proposition XI.21 involves a sequence of subsidiary results, mostly taken from Book I of the *The Elements*. Since the proposition itself seems, as do so many results in Euclid, almost obvious, I should say a few words of comment about the argument.

One's intuition about why XI.21 is true is based, presumably, on the nearly physical feeling involved in squashing the vertex flat.

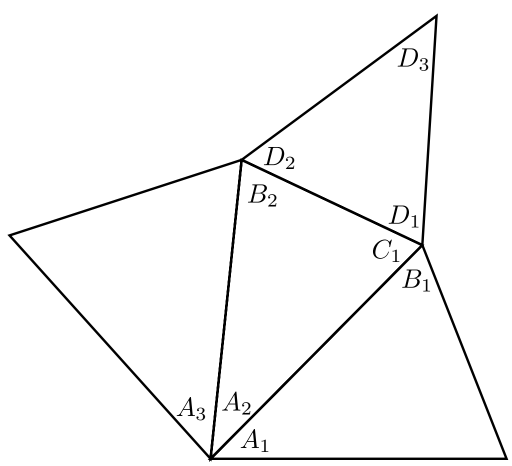


Translated into mathematics this is probably equivalent to the idea that projecting one of the facial angles of a convex vertex onto a plane spreads that angle out. And so it does, in the right circumstances, but not always. So it is not apparently true that we can make a direct comparison of the angles on each face with those in a 2D projection. Euclid must have been aware of this, although as usual he doesn't tell you more than you have to know. He manages, however, to get around the difficulties in a very elegant manner. I suppose his argument is a natural one, and one which some would perhaps call obvious. Nonetheless, I believe it to be one of the highlights in *The Elements*.

I shall present Euclid's argument by a backwards progression. First of all, along with Euclid I shall assume that the vertex is surrounded by three faces, in order to make the reasoning a little more concrete.



We need to label the figure. Cut off the faces by a plane Π intersecting them transversely. Each face becomes a triangle, and the interior of Π cut off is also a triangle. To picture better what is going on, we can unfold and spread these triangles out on a plane. Label the angles in these triangles like this:



Here A_1, A_2, A_3 are the angles immediately surrounding the original vertex. Thus, each one of the lower vertices
Proposition XI.20:6 in this tetrahedron will have angles B, C, D around it. This is the crucial fact:

- (Euclid XI.20) *If B, C, and D are any three angles around a trihedral vertex, then $B + C > D$.*

I'll postpone the proof of this for a moment, but right now let's see why this implies Proposition XI.21. In each triangle, the sum of its interior angles must be $180°$. Therefore

$$\sum (A_i + B_i + C_i) = 3 \cdot 180° \ .$$

where the sum is taken over the three faces of the vertex. But in addition, according to the result we have yet to prove

$$\sum (A_i + B_i + C_i) > \sum A_i + \sum D_i$$

$$\sum A_i < \sum (A_i + B_i + C_i) - \sum D_i$$

since $B_i + C_i > D_i$. However

$$\sum D_i = 180°$$

since the D_i are all the interior angles of a triangle. This gives us

$$\sum A_i < \sum (A_i + B_i + C_i) - \sum D_i = 3 \cdot 180° - 180° = 360° \ ,$$

which is just what Proposition XI.21 asserts.

Exercise 1. *Work out the proof for an arbitrary convex vertex.*

Suppose that, conversely, that one is given a collection of angles in the plane splayed out around a vertex, whose sum is less than $360°$. When can one construct a vertex in 3D that gives rise to it? Can one design an algorithm for doing this? First of all, this is not always possible. For example, for three angles α, β and γ with $\alpha+\beta+\gamma < 360°$ it is possible only if $\alpha + \beta > \gamma$, for example (as we shall see later on). And if there are more than three angles the vertex will not be unique—it will not in fact be **rigid**. This means that if one is given such a vertex that one can always move the faces around as movable plates without changing their shape. This is just another way of saying that polygons in the plane are similarly flexible—for example, one can always deform a square into a rhombus.

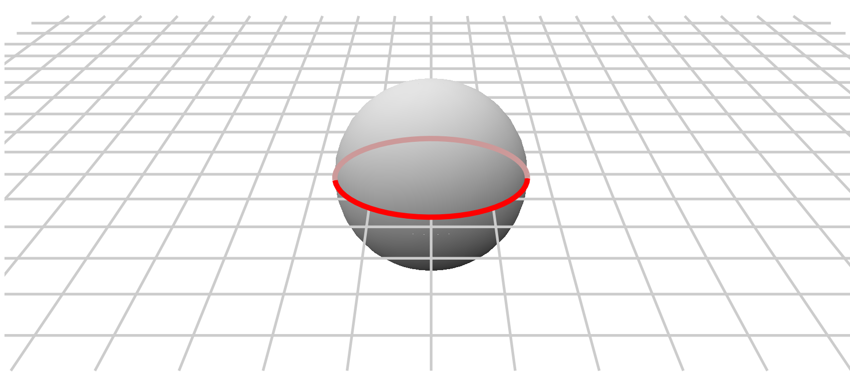
Exercise 2. *Suppose that $m\alpha < 360°$. Explain how to construct a regular vertex with vertex angle α—i.e. one whose orthogonal section is a regular polygon of m sides. (Hint: start with the regular polygon in a plane. The vertex should be somewhere on the perpendicular line through its centre. The fact that $m\alpha < 360°$ guarantees that the vertex can be located outide the plane.)*

4. Trihedral angles

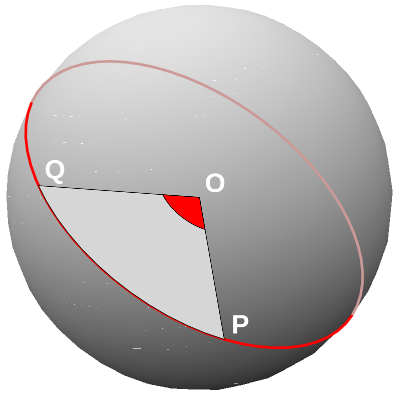
It remains to prove

- (Euclid XI.20) *If B, C, and D are any three angles around a trihedral vertex, then $B + C > D$.*

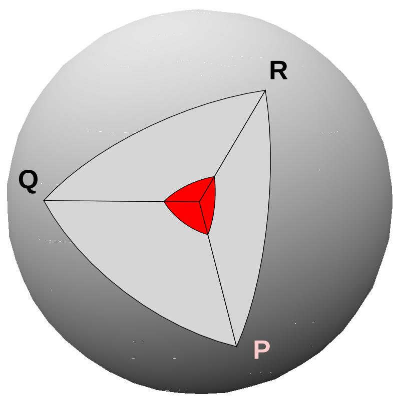
Before proving it, I will make it somewhat more plausible by translating it into a statement about geometry on a sphere. A **great circle** on a sphere is the intersection of the sphere with a plane through its origin.



Between any two points P and Q on a sphere which are not directly opposite to each other there passes a unique great circle, that determined by the plane containing P and Q and the sphere's centre O. The part of that great circle lying between the two points is the shortest route between them that lies entirely on the sphere.—the **spherical geodesic** line between them. Distance on the sphere along a great circle is proportional to the spanning angle POQ at the centre of the sphere. If the radius of the sphere is 1 then that distance is exactly equal to that angle measured in radians.



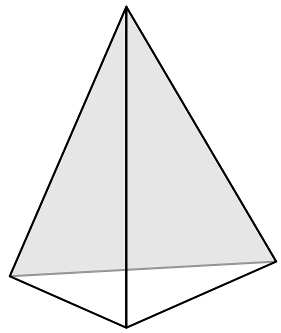
In particular, if R is a third point on the sphere which does not lie on the great circle arc between them, then the spherical distance PQ must be less than the sum of PR and RQ.

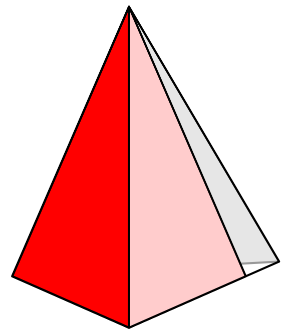


This is the *spherical triangle inequality*, analogous to the triangle inequality in the plane. Since spherical distances are proportional to central angles, *it is equivalent to the assertion we are trying to prove.* So in effect, in proving XI.21 we are proving the spherical analogue of a familiar inequality about paths on the plane (Euclid I.20, which we'll see in a moment).

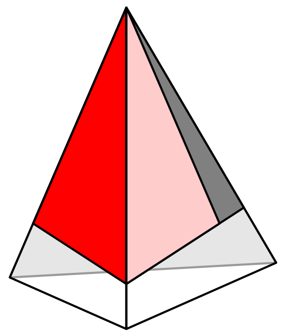
How about the proof itself?

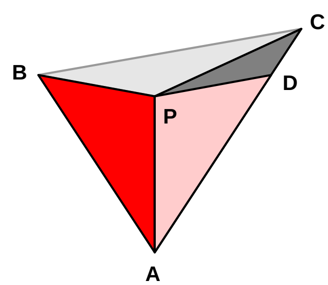
We follow Euclid. If all three of the vertex angles are the same, the claim is trivial. So suppose that one of the angles is actually larger than another. In the following figure at the left, these are on the faces we are looking at.





Lay a copy of the face with the smaller vertex angle on the face with the larger one, as in the figure on the right. Then slice off a triangle on the face with the larger vertex angle. Mirror that slice back onto the face with the smaller vertex angle, as in the figure on the left, just below:





Let P be the vertex, and label some other vertices as shown. The sum $AB + BC$ is greater than $AC = AD + DC$, by a result from Euclid (I.20) that I have already mentioned. Since $AD = AB$, $DC < BC$. By another result from Euclid (I.25), the angle DPC is less than the angle CPB. But then finally

s, Proposition I.25:9

$$APC = APD + DPC < APB + BPC \ .$$

Here are the results from Book I that we have used:

s, Proposition I.20:9

- (Euclid I.20) *In any triangle, the length of one side is less than the sum of the lengths of the two other sides.*
- (Euclid I.25) *If we are given two triangles two of whose sides match in length, then the angle opposite the third side is larger in the triangle with the longer third side.*

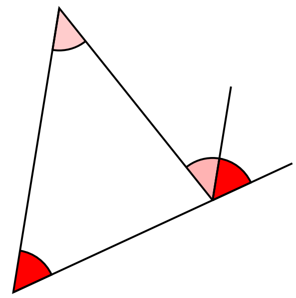
I'll recall their proofs in the next section.

5. The results we need from Book I

There are actually a number of results that we'll need from Book I of Euclid before we're through, since the proofs of the results we need will in turn take us back to others. Since we are not concerned here with complete rigour, but just with making the reasons as intuitively transparent as possible, the main difficulty is knowing where to begin. For some of these early results in Euclid, pictures alone should suffice.

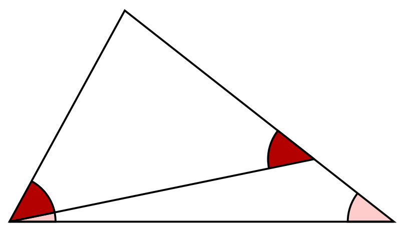
s, Proposition I.32:9

- (Euclid I.32) *In any triangle, the exterior angle of one corner is equal to the sum of the opposite interior angles.*



This will be applied in the next proof, but only in so far as it implies that the exterior angle is larger than either of the interior ones. This weaker result, unlike that above, can be proven without using the axiom of parallels, and , Proposition I.18:10 is Euclid I.16.

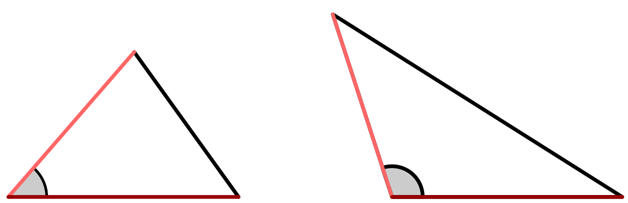
- (Euclid I.18) *In any triangle, the angle opposite a larger side is larger.*



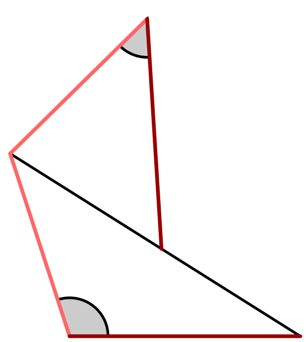
- (Euclid I.20) *The length of any side of a triangle is less than the sum of the other two sides.*

I leave this as an exercise, including pictures.

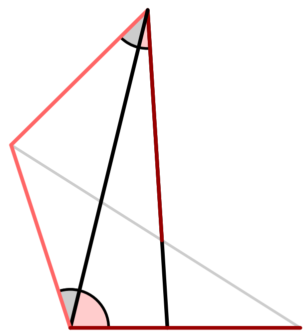
- (Euclid I.25) *Given two triangles with two sides in each matching two sides in the other, the one with the longer third side has the larger angle opposite the third side.*

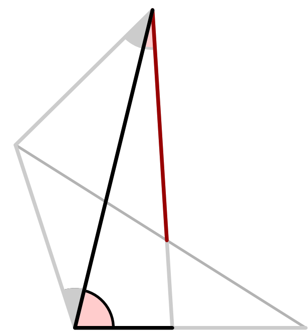


This we shall actually see proven. The demonstration I am about to give is attributed to Menelaus in Heath's comments on Proposition I.25. We start with the two triangles, pictured above. We translate the one with the smaller side and then rotate it and reflect it so as to get this picture:



Then we construct the isosceles triangle as shown below, and extend the line also. Finally, we apply the previous Proposition.





This concludes the proof that there are no more than 5 regular solids. It remains to tell how to construct them.

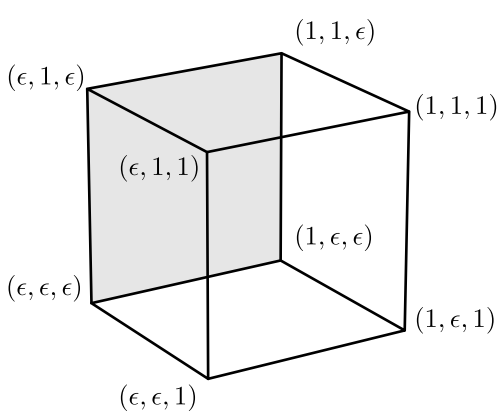
6. Constructing the regular polyhedra

As far as showing that the regular solids can be constructed is concerned, the proof above is somewhat limited in relevance. It says no more and no less than that *a single vertex of each of the regular polyhedra can be constructed.* But constructing a vertex is not the same as constructing the whole figure, since it is not at all obvious that the construction of a vertex can be extended to give the whole figure. Of course starting with one vertex we can go on building new vertices attached to what we already have, but there is no obvious reason why at some point we won't get some kind of peculiar incompatibility between pieces we have constructed. In fact, one can do a lot of calculations for each of the five regular solids and see that this problem does not arise. But an argument which shows directly and uniformly in all cases that such an incompatibility never occurs was found, as far as I know, only fairly recently. The argument we shall see here, following Euclid, looks at each case on its own. There is one notable accidental feature, however—it turns out that all of the regular polyhedra can be constructed by starting with cubes!

In the rest of this section I shall describe without proof the essentials of construction in all cases. In the next I shall sketch the justification of the construction.

• Cube

cube:11 This is easy. I make its side of length 2, aligned along the axes, with one corner at $(-1,-1,-1)$. Then the corners are all points with either 1 or $\epsilon = -1$ as coordinate, making eight in all.



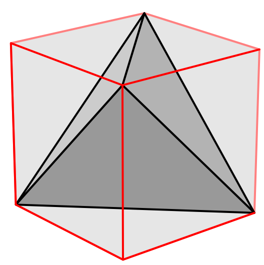
In the PostScript data file `regular-polyhedron.inc` describing the regular polyhedra, these points are put into an array:

```
[[-1 -1 -1]
[-1 1 -1]
[1 1 -1]
[1 -1 -1]
[-1 -1 1]
[-1 1 1]
[1 1 1]
[1 -1 1]]
```

That is to say, I go around the back square with $z = -1$ in the positive orientation *as seen from behind*, starting from the origin, then around the front face $z = 1$ in a parallel track.

• Tetrahedron

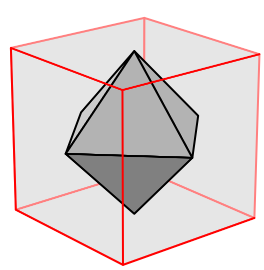
rahedron:regular:12 The vertices of the cube (x, y, z) with $x + y + z$ equal to -3 or 1 are the vertices of a regular tetrahedron, as are those where the sum is 3 or -1.



Exercise 3. *Prove this. Find an exact formula for the* **height** *of the tetrahedron, the distance from a vertex to the opposite face. Find the length of an edge.*

• Octahedron

tahedron:regular:12 The centres of the faces of a cube form a regular octahedron. The octahedron is therefore the **dual** of the cube.

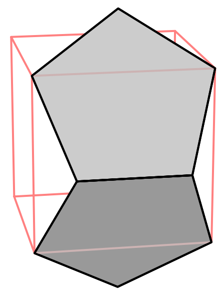


Exercise 4. *Find the length of an edge of this octahedron.*

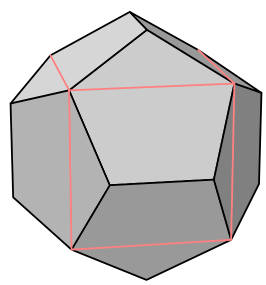
• Dodecahedron

cahedron:regular:12 The tetrahedron and octahedron are relatively simple figures. Their analogues and that of the cube exist in all dimensions. It is perhaps more surprising that a dodecahedron can also be constructed by starting with a cube.

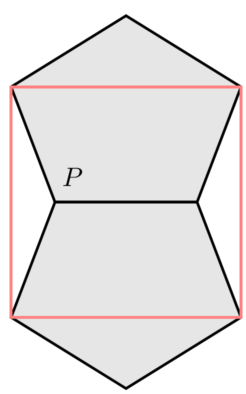
First construct a regular pentagon whose diagonal is equal to the side of the cube. Attach it along a diagonal to an edge of the cube, in effect making the diagonal into a hinge. Attach another congruent pentagon to the edge opposite, on the same face. You can check that if the two pentagons lie flat on the common face of the cube, they will overlap. If they are rotated away from the cube, of course eventually they will have no intersection. So somewhere in between they can be situated like this, so they just touch:



The remarkable thing is that you can attach a pair of pentagons to each face of the cube in this way, changing the orientation if necessary, so as to have twelve pentagons making up a dodecahedron with the 12 pentagons for faces.

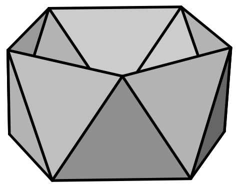


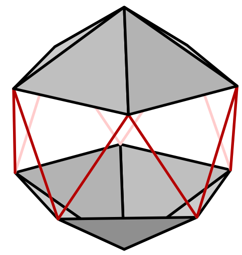
Exercise 5. *Find the coordinates* (x, y, z) *of the point* P *in 3D above the face of a cube making this work.*



• Icosahedron

sahedron:regular:13 The icosahedron is different. Assemble a band of ten equilateral triangles, and then add to this two pentagonal caps of five equilateral triangles.





The icosahedron can also be constructed as the dual of the dodecahedron.

Exercise 6. *Find the coordinates of all the vertices, and in particular the vertical height of the top vertex and the top pentagon.*

Exercise 7. *The icosahedron, too, can be constructed more directly from the cube. Write a PostScript program to follow the recipe of H. Taylor described in the comments on Proposition XIII.16 in Heath's English edition of Euclid.*

roposition XIII.16:14

7. Verifying regularity

The dodecahedron and icosahedron are not constructed uniformly, and it is not apparent that they are in fact regular. I leave this as an exercise.

For the dodecahedron, in addition to showing that the vertices are all congruent, it must be shown that the pentagons constructed on each face actually attach to the pentagons from other faces in the way they should.

For the icosahedron, the faces join together and are all congruent by definition. What remains to be shown is that the vertices are congruent.

8. Code

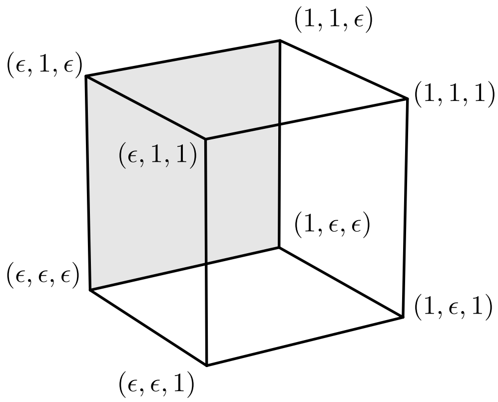
The regular polyhedra are catalogued in the file `regular-polyhedra.inc`. Their vertices are listed, and then their faces. Each face is an array of two items, first the the array of vertices on the face, traversed in a counter-clockwise direction, and then the coefficients `[A B C D]` such that $Ax + By + Cz + D \geq 0$ describes the outside of that face.

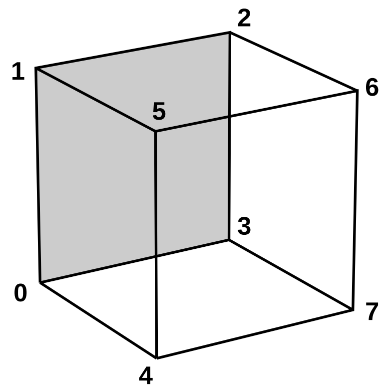
Here, for example, is the listing for the cube:

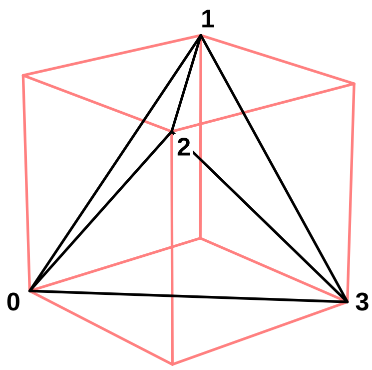
```
/cube-vertex [
  [-1 -1 -1]
  [-1 1 -1]
  [1 1 -1]
  [1 -1 -1]
  [-1 -1 1]
  [-1 1 1]
  [1 1 1]
  [1 -1 1]
] def

/cube [
```

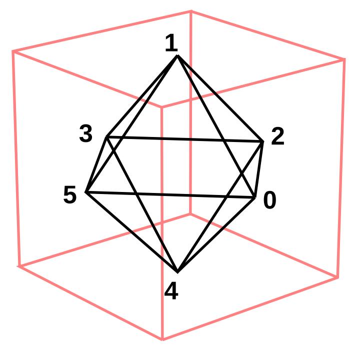
```
[
  [
    cube-vertex 0 get
    cube-vertex 1 get
    cube-vertex 2 get
    cube-vertex 3 get
  ] dup normal-function
]
[
  [
    cube-vertex 4 get
    cube-vertex 7 get
    cube-vertex 6 get
    cube-vertex 5 get
  ] dup normal-function
]
[
  [
    cube-vertex 0 get
    cube-vertex 4 get
    cube-vertex 5 get
    cube-vertex 1 get
  ] dup normal-function
]
[
  [
    cube-vertex 6 get
    cube-vertex 7 get
    cube-vertex 3 get
    cube-vertex 2 get
  ] dup normal-function
]
[
  [
    cube-vertex 2 get
    cube-vertex 1 get
    cube-vertex 5 get
    cube-vertex 6 get
  ] dup normal-function
]
[
  [
    cube-vertex 0 get
    cube-vertex 3 get
    cube-vertex 7 get
    cube-vertex 4 get
  ] dup normal-function
]
] def
```

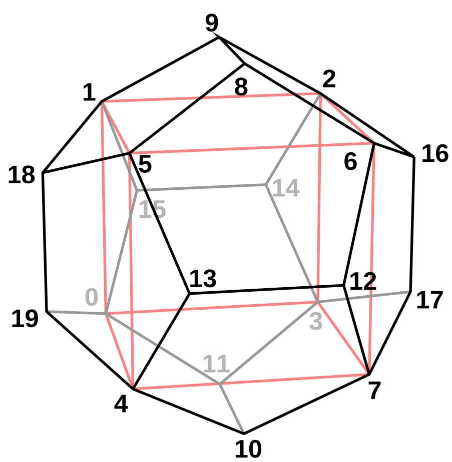
The file `regular-polyhedron.inc` contains enough data to describe all the regular polyhedra. There are commands `tetrahedron`, `octahedron`, `dodecahedron`, and `icosahedron` which return for each figure an array of faces like the one shown above for the cube. In order to use it, you have to know the numbering scheme for the vertices. Here are some diagrams which do that. We start off with one we have seen before.











Finally, I just mention that the numbering of the icosahedron starts at the top and goes down. With this information, it can be deduced from the facial structure.

9. References

Coxeter:H. S. M.:16 1. H. S. M. Coxeter, **Regular polytopes**, Dover. Regular polyhedra in all dimensions are dealt with in this classic. Euclid's construction of the five regular polyhedra is impressive, but there is something unsatisfying in that each is on its own. He does not present a uniform way to build all. The modern way to construct them is through their **symmetries**—the rigid transformations that take them into themselves—and that is what Coxeter's book is all about. The crucial point is that the symmetries are generated by reflections. Other situations in which this occurs are in the main stream of mathematics, and give rise among are associated with the most interesting of all mathematical objects. They have come to play a role in nearly branches of the field. The best construction of Tits:J.:16 these from simple data was discovered by Jacques Tits about 1960, and is contained in the surprisingly readable book **Groupes et algèbres de Lie IV, V, VI** by Nicholas Bourbaki.

Cromwell:P.:17 Cauchy:17 2. Peter Cromwell, **Polyhedra**, Cambridge University Press. Lots of stuff here. The rigidity theorem of Cauchy is in Chapter 6. Euclid's proposition that the sum of angles around a convex vertex is less than 360° was explained in a more geometric way—a precursor of curvature arguments—by Descartes, and this is in Chapter 5.

lements:Euclid's:17 3. Euclid, **The Elements**. All three volumes are in print from Dover, and the text of Euclid itself can be found at

`http://aleph0.clarku.edu/ djoyce/java/elements/elements.html`

Heath's comments on Euclid's text, which are extremely valuable, used to be available at the Perseus site

`http://www.perseus.tufts.edu/cgi-bin/ptext?lookup=Euc.+1`

but now (August, 2003), as I have mentioned elsewhere, only the text of Heath's translation is there, with comments exclusively from the introductory chapters. This is a great loss.

4. Jürgen Mau and Wolfgang Müller, 'Mathematische Ostraka aus der Berliner Sammlung', *Archiv für Papyrusforschung* **17** (1982), pp. 1–10. This is the only published account of the pottery shards found on Elephantine Island, containing the oldest extant fragments of Greek mathematics. They are concerned with technical details *The Elements*:17 of the construction of an icosahedron, related to material in Boox XIII of Euclid's *Elements*.