Chapter 1

Analytic Poissonization and Depoissonization

In combinatorics and the analysis of algorithms a Poisson version of a problem (henceforth called *Poisson model* or *poissonization*) is often easier to solve than the original one, which is usually known under the name *Bernoulli model*. Poissonization is a technique that replaces the original input by a Poisson process. Poisson transform maps a sequence characterizing the Bernoulli model into a generating function of a complex variable characterizing the Poisson model. Once the problem is solved in the Poisson domain, one must *depoissonize* it in order to translate the results back to the original (i.e., Bernoulli) model. A large part of this chapter is devoted to various depoissonization results. As a matter of fact, analytic depoissonization can be viewed as an asymptotic technique applied to generating functions that are entire functions. We illustrate our analysis on numerous examples from combinatorics and the analysis of algorithms and data structures. These applications are among the most sophisticated that we discuss in this book.

SOME ALGORITHMS (e.g., sorting and hashing) can be modeled by the so–called "balls-and-urns" paradigm in which n elements are placed randomly into m bins. Questions arise such as how many urns are empty, how many balls are required to fill up all urns, etc. It is easy to see that the occupancies of urns are not independent (e.g., if all balls fall into one urn, then all the remaining urns are empty). To overcome this difficulty an interesting probabilistic technique called **poissonization** was suggested (cf. Section 5.2). In this new Poisson model it is assumed that balls are "generated" according to a Poisson process N with mean n. Due to some unique properties of the Poisson process, discussed below, the streams of balls are now placed independently in every urn, thus overcoming the above mentioned difficulty. Observe, however, that poissonization has its own problem since one must "extract" the original results from the Poisson model, that is, **depoissonize** the Poisson model. Throughout this chapter we shall refer to the original model as the *Bernoulli model*.

More generally, let g_n be a characteristic of the Bernoulli model of size n (e.g., g_n is the number of empty urns in the balls-and-urns model with n urns). The **Poisson transform** is defined as

$$\widetilde{G}(z) = \mathbf{E}[g_N] = \sum_{n\geq 0} g_n \frac{z^n}{n!} e^{-z}.$$

Observe that the input n becomes a Poisson variable N with mean z when $z \geq 0$. We later extend z to the whole complex plane. The Poisson transform was introduced by Gonnet and Munro [174] and extended in [339]. When z is complex, we also refer to it as **analytic poissonization**.

To further motivate our discussion, let us consider the following scenario. We have already seen in this book that many problems arising in the analysis of algorithms are mapped into certain recurrence equations or functional/differential equations (e.g., Sections 7.6.1, 8.7.3 and 9.4.1). Algorithms involved in splitting processes and digital trees are representative examples of this situation. Embedding this splitting process into a Poisson process often leads to more tractable functional/differential equations. Take the following general recurrence describing a splitting process, already discussed in Section 7.6.1,

$$x_n = a_n + \beta \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} (x_k + x_{n-k}) , \qquad n \geq 2$$

with some initial values for x_0 and x_1, where a_n is a given sequence, and $\beta > 0$ is constant. Let $X(z)$ and $A(z)$ be the exponential generating functions of x_n and a_n, respectively. By (7.17) of Table 7.3 we find

$$X(z) = \beta X(zp)e^{zq} + \beta X(zq)e^{zp} + A(z) - x_0 - x_1 z.$$

To further simplify it, we consider the Poisson transform $\widetilde{X}(z) = X(z)e^{-z}$, and then

$$\widetilde{X}(z) = \beta \widetilde{X}(zp) + \beta \widetilde{X}(zq) + \widetilde{A}(z) - x_0 e^{-z} - x_1 z e^{-z} \; . \tag{1.1}$$

This is a linear additive functional equation that can be solved by consecutive iterations as shown in (7.67) of Section 7.6.1. But after solving it and eventually finding an asymptotic expansion of $\widetilde{X}(z)$, we still need a method to extract the coefficients x_n.

In general, if the Poisson transform $\widetilde{G}(z)$ can be computed exactly, then one can extract the coefficient $g_n = n!\,[z^n](\widetilde{G}(z)e^z)$. This is called *algebraic or exact depoissonization.* However, in most interesting situations (cf. Example 3) the Poisson transform $\widetilde{G}(z)$ satisfies a functional/differential equation like (10.1), that usually cannot be solved exactly. Nevertheless, one can find an asymptotic expansion of $\widetilde{G}(z)$ for $z \to \infty$ (e.g., using the Mellin transform discussed in the last chapter) on the real axis and can bound $\widetilde{G}(z)$ in the complex plane. Then one aims at finding an asymptotic expansion of g_n from the asymptotics of $\widetilde{G}(z)$. This we shall call **analytic depoissonization**. In Chapter 8 we discussed several methods to extract the coefficients g_n asymptotically as $n \to \infty$, from an asymptotic expansion of its generating function. In particular, the singularity analysis is designed to handle this situation when the analytic function is of moderate growth. Analytic depoissonization can be viewed as another asymptotic method, usually applied to functions of rapid growth.

Analytic depoissonization is well understood when the only singularities of the Poisson transform $\widetilde{G}(z)$ are poles or algebraic singularities. In this case, one can either apply Cauchy's residue theorem or the singularities analysis (cf. Section 8.3.2). The reader is referred to Gonnet and Munro [174], Poblete [337] and Poblete, Viola and Munro [339] for such analyses. A new situation arises when the Poisson transform $\widetilde{G}(z)$ is an entire function and consequently does not have any singularity in the complex plane. In this case, one can rely only on partial information regarding $\widetilde{G}(z)$ and apply the saddle point method (cf. Section 8.4) to extract the asymptotics. This is exactly what we plan to do in this chapter.

Finally, one may ask *why poissonization*? Why not embed the Bernoulli model into another process? This seems to be a consequence of certain unique properties of the Poisson process that we briefly discuss below. In short, it is the only process that has stationary and independent increments, and which has no group arrivals. To be a little more precise, let us first define *superposition* and *thinning* or *splitting* processes of a renewal (point) process (details on renewal processes can be found in Durrett [115] or Ross [365]). Consider two (stationary) renewal processes, say N_1 and N_2. Then the superposition process $N = N_1 + N_2$ consists of all renewal points of both processes. To define the splitting or thinning process, take a single renewal process and for each point decide independently whether to omit it (thinning) or to direct it to one of two (or more) outputs (splitting). The following three properties are well known (cf. [365]), but the reader is asked to prove them in Exercise 1:

(P1) A stationary renewal process is the superposition of some independent renewal processes *only if* the process is Poisson.

(P2) A thinning or splitting process is Poisson with parameter zp, where p is the probability of thinning if the original process is Poisson with parameter z.

(P3) Let $N(t)$ denote Poisson arriving points in the interval $(0,t)$. Then

$$\Pr\{N(x) = k \mid N(t) = n\} = \binom{n}{k}(x/t)^k(1-x/t)^{n-k},$$

where $x \leq t$. In other words, the points of Poisson process are uniformly and independently distributed in $(0,t)$ *conditioned* on n arrivals in this interval.

The last property is sometimes called "random occurrence of the conditional Poisson process".

This chapter discusses results obtained only recently that are still under vigorous investigations. Our goal is to present a readable account of poissonization and depoissonization. However, we also aspire to present rigorous derivations of depoissonization results so that this chapter can be used as a reliable reference on depoissonization. All depoissonization results presented in this chapter fall into the following general scheme: *If the Poisson transform has an appropriate growth in the complex plane, then an asymptotic expansion of the sequence can be expressed in terms of the Poisson transform and its derivatives evaluated on the real line.* Not surprisingly, actual formulations of depoissonization results depend on the nature of growth of the Poisson transform, and thus we have *polynomial* and *exponential* depoissonization theorems. Normalization (e.g., as in the central limit theorem) introduces another twist that led us to formulate the so–called *diagonal* depoissonization theorems. We also have a short account on the so–called *Dirichlet depoissonization* recently introduced by Flajolet (cf. [66]). In the application section we discuss the limiting distribution of the time to find a leader using the so–called leader election algorithm, and then we analyze the depth of b-digital search trees for memoryless and Markovian sources. The latter application is one of the most complex in this book.

To the best of our knowledge, poissonization was introduced by Marek Kac [224], who half a century ago investigated the deviations between theoretical and empirical distributions. Aldous [4] gave a heuristic probabilistic principle for depoissonizing small probabilities rather than large expected values which is main focus of this chapter. Exact analytic depoissonization results for meromorphic functions were discussed in Gonnet and Munro [173, 174] and Poblete [337], and Poblete, Viola and Munro [339]. Asymptotic analytic depoissonization for meromorphic function can be found in [174, 339] while depoissonization for entire functions was initiated by Jacquet and Régnier who analyzed the limiting distributions of the depth and the size in tries [210, 211, 352]. Jacquet and Régnier introduced the basic idea of the analytical depoissonization for an implicit solution of a non–linear functional equation. Jacquet and Szpankowski [215], and Rais *et al.* [346] extended these results and obtained the first general and simple version of the depoissonization result. The presentation in this chapter is based on two recent papers written together with my colleague P. Jacquet [216, 217]. We report here only basic depoissonization results and some of its generalizations. We refer the reader to Jacquet and Szpankowski [216] for more advanced depoissonization results. Recently, poissonization was further popularized in the context of analysis of algorithms and combinatorial structures by Aldous [4], Arratia and Tavaré [20], Clement, Flajolet and Vallée [66], Gonnet [173], Gonnet and Munro [174], Holst [197], Jacquet and Régnier [210, 211], Jacquet and Szpankowski [212, 215, 216, 217], Janson and Szpankowski [221], Rais *et al.* [346], Fill *et al.* [122], Kirschenhofer *et al.* [252], Poblete [337], and Poblete *et al.* [339].

Finally, we must observe that any depoissonization result is in fact a Tauberian theorem for the Borel mean(cf. [186]), which is nothing else but the Poisson transform defined above. For an accessible account on modern development of Tauberian theorems the reader is referred to Bingham [48].

1.1 Analytic Poissonization and the Poisson Transform

We briefly discuss the poissonization technique. First, we review probabilistic poissonization, then discuss exact analytic poissonization, and finish this section with an overview of asymptotic poissonization.

1.1.1 Poisson Transform

Consider a combinatorial structure in which n objects are randomly distributed into some locations (e.g., one can think of n balls thrown into urns). The objects are not necessarily uniformly distributed among the locations (cf. digital trees). We call such a setting the Bernoulli model. Let X_n be a characteristic of the model (e.g., the number of throws needed to fill up all urns). Next, we define the Poisson model. Let N be a random variable distributed according to a Poisson distribution with parameter $z \geq 0$, that is, $\Pr\{N = k\} = e^{-z}z^k/k!$, and let X_N be the above characteristic defined in the Poisson model in which the *deterministic* input (i.e., n) is replaced by the Poisson variable N. Then by definition

$$\begin{aligned} \widetilde{X}(z) := \mathbf{E}[X_N] &= \sum_{n\geq 0} \mathbf{E}\left[X_N \mid N = n\right] e^{-z}\frac{z^n}{n!} \\ &= \sum_{n\geq 0} \mathbf{E}[X_n]e^{-z}\frac{z^n}{n!}. \end{aligned} \tag{1.2}$$

We now analytically continue $\widetilde{X}(z)$ to the whole complex plane, and throughout we make the following assumption:

(A) The sum in (10.2) converges absolutely for every z, that is, $\widetilde{X}(z)$ is an entire function of the complex variable z.

This defines the Poisson transform $\widetilde{X}(z)$ for all complex z.

In general, we use g_n as a generic notation for a sequence characterizing the Bernoulli model (e.g., $g_n = \mathbf{E}[X_n]$ or, in general, $g_n = \mathbf{E}[f(X_n)]$ for some function f). By $\widetilde{G}(z)$ or $\mathcal{P}(g_n; z)$ we denote the **Poisson transform** defined formally as

$$\widetilde{G}(z) = \mathcal{P}(g_n; z) := \sum_{n\geq 0} g_n \frac{z^n}{n!}e^{-z} \ .$$

Table 10.1 presents some common Poisson transforms. Items 1-4 are obvious. Items 5 (the so–called additive (p, q)–splitting, where $q = 1 - p$) and Item 6 (i.e., multiplicative (p, q)–splitting) follow directly from (7.17) of Table 7.3, while Item 7 requires only trivial algebra (we return to it in Section 10.3.1). A more complete list of Poisson transforms can be found in [339].

An astute reader should notice that for Items 1 and 2 of Table 10.1 $g_n \sim \widetilde{G}(n)$ as $n \to \infty$ while this is not true for Items 3 and 4. In the latter case, the Poisson transform is either of rapid growth (Item 3) or not an entire function (so assumption (A) is not satisfied). For Items 1 and 2 the transforms are of a polynomial growth. This is not a coincidence, and in this chapter we systematically explore the relationship between g_n as $n \to \infty$ and $\widetilde{G}(z)$ as $z \to \infty$.

Table 1.1: Some Poisson Transforms

Item	g_n	$\widetilde{G}(z)$
1.	C	C
2.	$n(n-1)\cdots(n-k+1)$	z^k
3.	α^n	$e^{(\alpha-1)z}$
4.	$n!$	$\frac{e^{-z}}{1-z}$
5.	$\sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}(f_k+g_{n-k})$	$F(pz)+G((1-p)z)$
6.	$\sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}f_k g_{n-k}$	$F(zp)G((1-p)z)$
7.	$\sum_{k\geq 1}a_k(1-(1-a_k)^n)$	$\sum_{k\geq 1}a_k(1-e^{-za_k})$

Example 1.1 *Depth and Size of a Trie*

Let us find the Poisson transform of the depth D_n for a trie built over n strings generated by biased memoryless source. We define

$$\begin{aligned}
D_n(u) &= \mathbf{E}[u^{D_n}] = \sum_{k=0}^{\infty}\mathsf{Pr}\{D_n=k\}u^k, \\
D(z,u) &= \sum_{n=0}^{\infty}\mathbf{E}[u^{D_n}]\frac{z^n}{n!}, \\
\widetilde{D}(z,u) &= e^{-z}D(z,u).
\end{aligned}$$

We now derive a functional equation for $\widetilde{D}(z,u)$ directly from the properties of tries and the Poisson process (without going through a recurrence equation on D_n). We claim that

$$\widetilde{D}(z,u) = u(p\widetilde{D}(zp,u)+q\widetilde{D}(zq,u)) + (1-u)(1+z)e^{-z}.$$

To see this, we first observe that digital trees are recursive structures, so the Poisson transform of the depth of the left subtree is $D(zp,u)$ while for the right subtree is $D(zq,u)$ (by property (P2) of the Poisson process). Since the subtrees are one level lower than the root, we have the factor u. But when computing the depth we either go left (with probability p) or right (with probability q), but never both ways. Finally, we know that $n=0$ and $n=1$ are special cases since for such n there is no left/right subtree recurrence. The probability of having $n=0$ or $n=1$ in the Poisson model is equal to $(1+z)e^{-z}$. We must subtract from it $u(1+z)e^{-z}$ which represents the subtree structures not accounted in the recurrence. This establishes the above functional equation.

In Exercise 2 the reader is asked to prove that the Poisson transform $\widetilde{S}(z,u)$ of the trie size satisfies

$$\widetilde{S}(z,u) = u\widetilde{S}(zp,u)\widetilde{S}(zq,u) + (1-u)(1+z)e^{-z}$$

for $|u| \le 1$. □

1.1.2 Asymptotic Poissonization

In the previous section we computed the Poisson transform exactly. But there are sequences (e.g., $g_n = \log n$ and $g_n = n^\alpha$ for non-integer α) for which $\widetilde{G}(z)$ cannot be found in a closed-form. The natural approach then is to compute $\widetilde{G}(z)$ asymptotically. We search for the asymptotics of $\widetilde{G}(z)$ as $z \to \infty$ in a linear cone

$$\mathcal{S}_\theta = \{z : \ |\arg(z)| \le \theta,\ 0 < \theta < \pi/2\},$$

around the positive real axis. One expects to find asymptotic expansion $\widetilde{G}(z)$ in the following form

$$\widetilde{G}(z) = \sum_{i,j\ge 0} a_{ij} z^i g^{(j)}(z), \qquad z \to \infty,$$

where $g^{(j)}(z)$ is the jth derivative of $g(z)$ which is the analytic continuation of g_n, if it exists, that is, $g_n = g(z)|_{z=n}$.

We first identify the coefficients a_{ij} formally, and later derive an error bound. We proceed as follows:

$$\begin{aligned}\widetilde{G}(z) &= \sum_{n=0}^{\infty} g(n)\frac{z^n}{n!}e^{-z} = \sum_{n=0}^{\infty}\frac{z^n}{n!}e^{-z}\sum_{j=0}^{\infty} g^{(j)}(z)\frac{(n-z)^j}{j!} \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} a_{nj} z^n g^{(j)}(z). \end{aligned} \tag{1.3}$$

We would like to formally set

$$a_{nj} = [z^n][y^j]\frac{1}{n!}\frac{(n-z)^j}{j!}e^{-z},$$

where y^j represents formally $g^{(j)}(z)$. This seemingly strange substitution becomes more familiar, if we consider the following series

$$\begin{aligned}\sum_{n,j\ge 0} a_{n,j} z^n y^j &= \sum_{n=0}^{\infty} e^{-z}\frac{z^n}{n!}\sum_{j=0}^{\infty}\frac{(n-z)^j}{j!}y^j = \sum_{n=0}^{\infty} e^{-z}\frac{z^n}{n!}e^{y(n-z)} \\ &= \exp\left(z(e^y-1) - zy\right).\end{aligned}$$

Thus formally we define the coefficients a_{ij} as

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}x^i y^j = \exp(x(e^y-1)-xy), \tag{1.4}$$

that is, $a_{ij} = [x^i][y^j]\exp(x(e^y-1)-xy)$. Actually, $a_{ij}=0$ for $j<2i$. To see this, let $f(x,y)=\exp\left(x(e^y-1)-xy\right)$. Observe that $f(xy^{-2},y)$ is analytic at $x=y=0$, hence the Laurent series of $f(x,y)$ possesses only terms like $x^i y^{j-2i}$ with *nonnegative* powers. These nonzero coefficients a_{ij} have $j\geq 2i$.

In view of this, we can write formally $\widetilde{G}(z)$ in short as

$$\widetilde{G}(z) = \exp\left(z(e^y-1)-zy\right), \qquad \text{where} \qquad y^j = g^{(j)}(z).$$

This is a convenient way of expressing analytic poissonization.

In some applications we really want to have an expansion around $w=\lambda z$, where λ is constant, so we extend a_{ij} to $a_{ij}(\lambda)$ defined as

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}(\lambda)x^i y^j = \exp(\lambda x(e^y-1)-\lambda xy).$$

Observe that $a_{ij}(\lambda)=\lambda^i a_{ij}$. The same derivation as above leads to the following symbolic representation for $\widetilde{G}(z)$

$$\widetilde{G}(w) = \exp\left(w(e^y-1)-wy\right) \qquad \text{where} \qquad y^j = g^{(j)}(w) \tag{1.5}$$

and $w=\lambda z$ with λ being a constant. In addition, we directly obtain derivatives of $\widetilde{G}(w)$ as

$$\widetilde{G}^{(k)}(w) = (e^y-1)^k\exp\left(w(e^y-1)-wy\right) \qquad \text{where} \qquad y^j = g^{(j)}(w)\ . \tag{1.6}$$

So far we have used a formal series approach to find $\widetilde{G}(z)$. But we are really interested in an asymptotic expansion of $\widetilde{G}(z)$ as $z\to\infty$, thus we must derive an error bound. This is not difficult. However, one must first bound derivatives of an analytic function, and this is discussed next. This is of prime importance to the depoissonization results studied in the next section.

Lemma 1.1 *Let $\theta_0<\pi/2$ and $D>0$ be such that for all $z\in\mathcal{S}_{\theta_0}$ the function $G(z)$ is analytic in this cone and*

$$|z|>D \qquad \Longrightarrow \qquad |G(z)|\leq B|z|^{\beta} \tag{1.7}$$

for some reals β and $B>0$. Then for all $\theta<\theta_0$ there exist $B'>0$ and $D'>D$ such that for all positive integers k and $z\in\mathcal{S}_\theta$ the following holds

$$|z|>D' \qquad \Longrightarrow \qquad |G^{(k)}(z)|\leq k!(B')^k|z|^{\beta-k}\ . \tag{1.8}$$

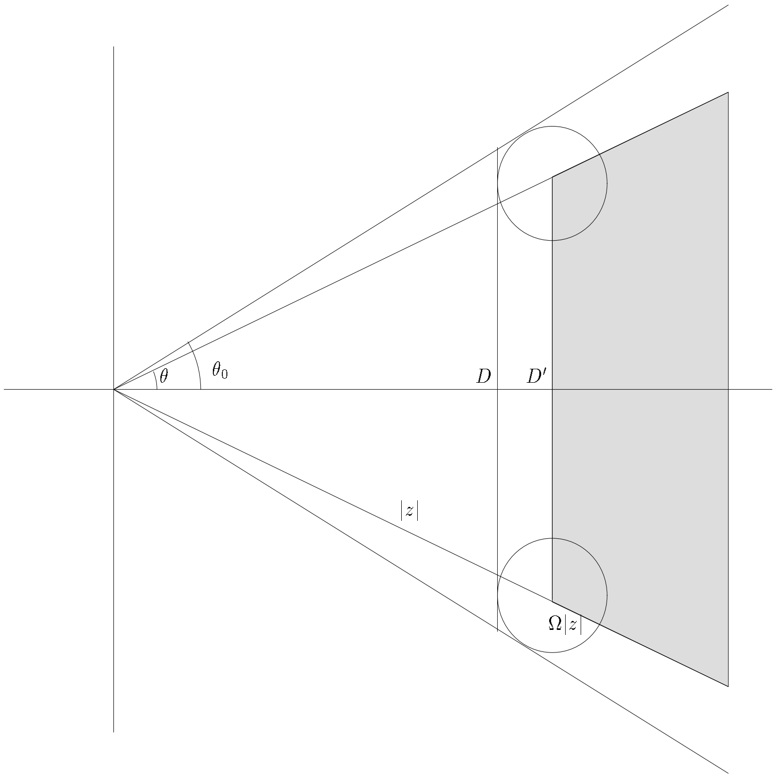

Figure 1.1: Illustration to Lemma 10.1.

Proof. The proof follows directly from the Cauchy integral formula applied to derivatives. Indeed, from Cauchy's formula we know that

$$G^{(k)}(z_0) = \frac{k!}{2\pi i} \oint \frac{G(z)dz}{(z-z_0)^{k+1}} \tag{1.9}$$

for $z \in B(z_0, r) = \{z : |z - z_0| \leq r\} \subset \mathcal{S}_\theta$, assuming $G(z)$ is analytic in $B(z_0, r)$ for some $r > 0$. If $|G(z)| \leq M$ for some $M > 0$, then clearly the above implies

$$|G^{(k)}(z)| \leq Mk!r^{-k} \tag{1.10}$$

for $z \in B(z_0, r)$. To prove our lemma, we place a circle of radius $r = \Omega|z|$ for some $\Omega > 0$ at the closest corner of the cone $\mathcal{S}_{\theta_0}$ to the right of $\Re(z) = D$ as shown in Figure 10.1. Setting $M = B|z|^\beta$ in (10.10) we prove Lemma 10.1 with $B' = B\Omega^{-k}$. ∎

Now we can formulate our main asymptotic poissonization result.

Theorem 1.2 (Jacquet and Szpankowski, 1999) *Let $g(z)$ be an analytic continuation of $g(n) = g_n$ such that $g(z) = O(z^\beta)$ in a linear cone for some constant β. Then for every nonnegative integer m and complex $w = \lambda z$ for some constant λ the Poisson transform $\widetilde{G}(z)$ as $z \to \infty$ has the following representation*

$$\begin{aligned} \widetilde{G}(z) &= \sum_{i=0}^{m} \sum_{j=0}^{i+m} a_{ij}(\lambda) z^i g^{(j)}(w) + O(z^{\beta-m-1}) \qquad (1.11) \\ &= g(w) + \sum_{k=1}^{m} \sum_{i=1}^{k} a_{i,k+i}(\lambda) z^i g^{(k+i)}(w) + O(z^{\beta-m-1}), \end{aligned}$$

where

$$a_{ij}(\lambda) = [x^i][y^j] \exp(\lambda x(e^y - 1) - \lambda xy).$$

In particular, for $\lambda = 1$

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} x^i y^j = \exp(x(e^y - 1) - xy) \qquad (1.12)$$

and $a_{ij} = 0$ for $j < 2i$.

Proof. In the view of our previous discussion, it suffices to derive the error term. Using Lemma 10.1 we observe that $g^{(j)}(z) = O(z^{\beta-j})$ and using $a_{ij} = 0$ for $j < 2i$ we arrive at the following error term

$$\sum_{j=0}^{2m+1} a_{m+1,j}(\lambda) z^m g^{(j)}(w) = O(z^{\beta-m-1})$$

as $z \to \infty$, since $w = \lambda z$. This completes the proof. ∎

Here are the first few terms of the expansion of $\widetilde{G}(z)$:

$$\begin{aligned} \widetilde{G}(z) &= g(z) + \frac{1}{2} z\, g^{(2)}(z) + \frac{1}{6} z\, g^{(3)}(z) + \left(\frac{1}{24} z + \frac{1}{8} z^2\right) g^{(4)}(z) \\ &+ \left(\frac{1}{120} z + \frac{1}{12} z^2\right) g^{(5)}(z) + \left(\frac{1}{720} z + \frac{5}{144} z^2 + \frac{1}{48} z^3\right) g^{(6)}(z) \\ &+ \left(\frac{1}{5040} z + \frac{1}{90} z^2 + \frac{1}{48} z^3\right) g^{(7)}(z) + \cdots. \end{aligned}$$

If one wants the error term of order, say $O(z^{\beta-4})$ (we assume $g(z) = O(z^\beta)$), then we need only one line of the above expansion plus selected terms from the second line, that is,

$$\begin{aligned} \widetilde{G}(z) &= g(z) + \frac{1}{2} z\, g^{(2)}(z) + \frac{1}{6} z\, g^{(3)}(z) + \left(\frac{1}{24} z + \frac{1}{8} z^2\right) g^{(4)}(z) \\ &+ \frac{1}{12} z^2 g^{(5)}(z) + \frac{1}{48} z^3 g^{(6)}(z) + O(z^{\beta-4}). \end{aligned}$$

Example 1.2 *Poisson Transform of* $f_n = \log(n!)$

Let us consider $f_n = \log(n!)$. This sequence arises in the analysis of the entropy of the binomial distribution (cf. Example 7 below). Noting that $f(z) = \log\Gamma(z+1) = O(z\log z) = O(z^{1+\varepsilon})$, from Theorem 10.2 we obtain for any $\varepsilon > 0$

$$\begin{aligned}\widetilde{F}(z) &= \sum_{i=0}^{m}\sum_{j=0}^{i+m} a_{ij}(\lambda) z^i f^{(j)}(w) + O(z^{\beta-m-1}) \\ &= f(z) + \frac{1}{2} z f^{(2)}(z) + \frac{1}{6} z f^{(3)}(z) + \frac{1}{8} f^{(4)}(z) + O(z^{\varepsilon-2}), \qquad m = 2.\end{aligned}$$

To find the derivatives of $f(z) = \log\Gamma(z) = \log z + \log\Gamma(z)$ we use the following expansions as $z \to \infty$:

$$\begin{aligned}f(z) &\sim \left(z+\frac{1}{2}\right)\ln z - z\frac{1}{2} + \sum_{m=1}^{\infty}\frac{B_{2m}}{2m(2m-1)z^{2m-1}} \\ f^{(k)}(z) &\sim \Psi^{(k-1)}(z+1) = \frac{(-1)^k(k-2)!}{z^{k-1}} + \frac{(-1)^{k-1}(k-1)!(2k-1)}{2z^k} \qquad \text{for } k \geq 2 \\ &\qquad + (-1)^k \sum_{l=1}^{\infty}\frac{B_{2l}(2l+k-2)!}{(2l)!\; z^{2l+k-1}},\end{aligned}$$

where $\Psi(z) = \frac{d}{dz}\log\Gamma(z)$ is the psi function and B_n are Bernoulli numbers (cf. Table 8.1). □

1.2 Basic Depoissonization

We consider now a sequence g_n and its Poisson transform $\widetilde{G}(z) = \mathcal{P}(g_n, z)$. Throughout, we assume that $\widetilde{G}(z)$ is an entire function. We first discuss exact or algebraic depoissonization in which the coefficients g_n are extracted exactly from its Poisson transform. Next, we focus on asymptotic depoissonization which is the main goal of this chapter. We again use Cauchy's formula to find

$$g_n = \frac{n!}{2\pi i}\oint \frac{\widetilde{G}(z)e^z}{z^{n+1}}dz = \frac{n!}{n^n 2\pi}\int_{-\pi}^{\pi}\widetilde{G}(ne^{it})\exp\left(ne^{it}\right)e^{-nit}dt\ . \tag{1.13}$$

All the asymptotic depoissonization results will follow from the above by a careful estimation of the integral using the saddle point method (cf. Section 8.4). Here we present, and give a simple proof, of a basic asymptotic depoissonization result. Later in Section 10.3 we generalize it.

1.2.1 Algebraic Depoissonization

The exact or algebraic depoissonization is based on the exact extraction of g_n from its Poisson transform, that is,

$$g_n = n![z^n]\left(e^z\widetilde{G}(z)\right).$$

In some cases this is quite simple. For example, applying directly Cauchy's formula to $\widetilde{G}(z) = e^{-z}/(1-z)$ (cf. Item 4 in Table 10.1) we find $n![z^n]\left(e^z\widetilde{G}(z)\right) = n!$, as required. The example below illustrates a more sophisticated case. The reader is referred to Gonnet and Munro [174], Poblete, Viola and Munro [339] and Hofri [195] for more interesting examples.

Example 1.3 *Ball-and-Urn Model*
Consider n balls (items) thrown randomly and uniformly into m urns (table of size m). What is the probability that precisely k specified urns are empty? Under the Poisson model the answer is immediate. Indeed, consider the Poisson process of balls with mean z. It is split into m *independent* Poisson processes of mean z/m entering each of m urns. All these processes are independent so that the probability $P_k(z)$ that exactly k specified urns are empty in the Poisson model is obviously equal to

$$P_k(z) = e^{-kz/m}(1-e^{-z/m})^{m-k} = e^{-z}(e^{z/m}-1)^{m-k}.$$

To find the answer to the original problem, we depoissonize $P_k(z)$ exactly, yielding

$$\begin{aligned}\Pr\{\text{number of empty urns} = k\} &= n![z^n]\left(e^z P_k(z)\right) = [z^n]\left(n!(e^{z/m}-1)^{m-k}\right)\\ &= \frac{(m-k)!}{m^n}\left\{ {n \atop m-k} \right\},\end{aligned}$$

where $\left\{ {n \atop k} \right\}$ denote the Stirling numbers of the second kind. The last line follows from Entry 9 of Table 7.2. The above formula can be obtained directly from the Bernoulli model. However, the derivation is much more troublesome. □

The situation encountered in the last example is rare. We usually cannot invert exactly the Poisson transform and we must resort to asymptotics. The rest of this chapter is devoted to asymptotic depoissonization. We start with a basic depoissonization (Section 10.2.2), then discuss in Section 10.3 some generalizations, and in Section 10.4 we deal with limiting distributions through depoissonization.

1.2.2 Asymptotic Depoissonization

Before we enter the realm of rigorous analysis, the reader should develop an intuition how depoissonization works. We first propose a heuristic derivation. We refer to the definition of the Poisson transform, which we write as $\widetilde{G}(z) = \mathbf{E}[g_N]$, where N is a Poisson distributed random variable with mean $\mathbf{E}[N] = n$. Treating g_N as a function of N (and denoting it as $g(N)$), by Taylor's expansion we have

$$g(N) = g(n) + (N-n)g'(n) + \frac{1}{2}g''(n)(N-n)^2 + \cdots.$$

Taking the expectation we obtain

$$\widetilde{G}(n) = \widetilde{G}(z)|_{z=n} = \mathbf{E}[g(N)] = g(n) + \frac{1}{2}g''(n)n + \cdots$$

since $\mathbf{E}[N-n] = 0$ and $\mathbf{E}[N-n]^2 = n$. Solving the above for $g(n) = g_n$ we find that

$$g_n \approx \widetilde{G}(n) - \frac{1}{2}ng''(n) + \cdots = \widetilde{G}(n) + O(ng''(n)). \tag{1.14}$$

Provided that

$$ng''(n) = o(g(n)), \tag{1.15}$$

we expect to have $g_n \sim \widetilde{G}(n)$. To emphasize this, consider the following examples:

- Let $g(n) = n^\beta$. Then $\widetilde{G}(n) = n^\beta + O(n^{\beta-1})$, $g''(n) = O(n^{\beta-2})$, thus

  $$g_n = \widetilde{G}(n) + O(ng''(n)) = \widetilde{G}(n) + O(n^{\beta-1}),$$

  which is true, as already seen in Table 10.1. This is proved in Theorem 10.3.

- Consider now $g(n) = \alpha^n$. This time $\widetilde{G}(z) = e^{z(\alpha-1)}$, $g''(n) = \alpha^n \log^2 \alpha$, and it is *not* true that $g_n \sim \widetilde{G}(n)$, as observed in Section 10.1.

- Now we assume $g_n = e^{n^\beta}$. In this case, it is harder to find the Poisson transform, but an extension of Theorem 10.2 suggests that $\widetilde{G}(z) \sim e^{z^\beta}$. We also have $g''(n) = O(n^{2\beta-2}e^{n^\beta})$. Observe that

  $$g_n = \widetilde{G}(n) + O(ng''(n)) = \widetilde{G}(n) + O(n^{2\beta-1}e^{n^\beta})$$

  and the error is small as long as $0 < \beta < \frac{1}{2}$. This is proved rigorously in Theorem 10.8.

The above derivation is hard to make rigorous since it is difficult to control the growth of the derivatives of a function while knowing only its behavior on the real axis. To circumvent this problem, we extend $\widetilde{G}(z)$ to the complex plane, where the growth of $g(z)$ and $\widetilde{G}(z)$ are easier to bound (cf. Lemma 10.1).

Enriched by the above intuition, we now prove rigorously some depoissonization results. We start with a basic depoissonization that holds for sequences having a Poisson transform of polynomial growth. As a matter of fact, we have to put certain restrictions on the growth of $\widetilde{G}(z)$ inside a cone $\mathcal{S}_\theta$ (called conditions (I)) and another bound on the growth of $\widetilde{G}(z)$ outside the cone (called conditions (O)). The proof presented below will be modified throughout this chapter to obtain further generalizations but its main ingredients remain the same. The reader is advised to invest some time to study the proof of the theorem below.

Theorem 1.3 (Jacquet and Régnier, 1987; Jacquet and Szpankowski, 1998) *Let $\widetilde{G}(z)$ be the Poisson transform of a sequence g_n that is assumed to be an entire function of z. We postulate that in a linear cone $\mathcal{S}_\theta$ ($\theta < \pi/2$) the following two conditions simultaneously hold:*
(I) *For $z \in \mathcal{S}_\theta$ and some reals $B, R > 0, \beta$*

$$|z| > R \quad \Rightarrow \quad |\widetilde{G}(z)| \le B|z|^\beta \ , \tag{1.16}$$

(O) *For $z \notin \mathcal{S}_\theta$ and $A, \alpha < 1$*

$$|z| > R \quad \Rightarrow \quad |\widetilde{G}(z)e^z| \le A \exp(\alpha|z|) \ . \tag{1.17}$$

Then

$$g_n = \widetilde{G}(n) + O(n^{\beta-1}) \tag{1.18}$$

for large n.

Proof. The proof relies on the evaluation of Cauchy's formula (10.13) by the saddle point method. By Stirling's approximation $n! = n^n e^{-n}\sqrt{2\pi n}(1 + O(1/n))$, thus (10.13) becomes

$$\begin{aligned} g_n &= \left(1 + O(n^{-1})\right)\sqrt{\frac{n}{2\pi}}\int_{-\pi}^{\pi} \widetilde{G}(ne^{it})\exp\left(n\left(e^{it} - 1 - it\right)\right) dt \\ &= \left(1 + O(n^{-1})\right)(I_n + E_n), \end{aligned}$$

where

$$\begin{aligned} I_n &= \sqrt{\frac{n}{2\pi}}\int_{-\theta}^{\theta} \widetilde{G}(ne^{it})\exp\left(n\left(e^{it} - 1 - it\right)\right) dt, \\ E_n &= \sqrt{\frac{n}{2\pi}}\int_{|t|\in[\theta,\pi]} \widetilde{G}(ne^{it})\exp\left(n\left(e^{it} - 1 - it\right)\right) dt \\ &= \frac{n^n e^{-n}\sqrt{2\pi n}}{2\pi i}\int_{|t|\in[\theta,\pi]} \frac{\widetilde{G}(z)e^z}{z^{n+1}} dz \ . \end{aligned}$$

We estimate the above two integrals. We begin with the latter. Observe that by condition (O) (cf. (10.17)), we obtain

$$|E_n| \le A'\sqrt{2\pi n}e^{-(1-\alpha)n},$$

where A' depends only on A and R. Thus E_n decays exponentially to zero as $n \to \infty$, since $\alpha < 1$.

Now we turn our attention to the integral I_n, which is more intricate to handle. First of all, we replace t by $t/\sqrt{n}$ to find that

$$I_n = \frac{1}{\sqrt{2\pi}} \int_{-\theta\sqrt{n}}^{\theta\sqrt{n}} \widetilde{G}(ne^{it/\sqrt{n}}) \exp\left(n\left(e^{it/\sqrt{n}} - 1 - it/\sqrt{n}\right)\right) dt \ . \tag{1.19}$$

Let

$$h_n(t) = \exp\left(n\left(e^{it/\sqrt{n}} - 1 - it/\sqrt{n}\right)\right) .$$

We need to estimate $h_n(t)$ in the interval $t \in [-\theta\sqrt{n}, \theta\sqrt{n}]$, and find the Taylor expansion of it in a smaller interval, say for $t \in [-\log n, \log n]$ (in order to apply Taylor's expansion of $h_n(t)$ around $t = 0$). The latter restriction is necessary since $\frac{t}{\sqrt{n}} = O(1)$ for $t \in [-\theta\sqrt{n}, \theta\sqrt{n}]$. Thus we split the integral I_n into two parts, I_n' and I_n'', such that

$$\begin{aligned}
I_n' &= \frac{1}{\sqrt{2\pi}} \int_{-\log n}^{\log n} \widetilde{G}(ne^{it/\sqrt{n}}) \exp\left(n\left(e^{it/\sqrt{n}} - 1 - it/\sqrt{n}\right)\right) dt \ , \\
I_n'' &= \frac{1}{\sqrt{2\pi}} \int_{t\in[-\theta\sqrt{n}, -\log n]} \widetilde{G}(ne^{it/\sqrt{n}}) \exp\left(n\left(e^{it/\sqrt{n}} - 1 - it/\sqrt{n}\right)\right) dt + \\
&+ \frac{1}{\sqrt{2\pi}} \int_{t\in[\log n, \theta\sqrt{n}]} \widetilde{G}(ne^{it/\sqrt{n}}) \exp\left(n\left(e^{it/\sqrt{n}} - 1 - it/\sqrt{n}\right)\right) dt \ .
\end{aligned}$$

To estimate the second integral I_n'' we observe that $|h_n(t)| \le e^{-\mu t^2}$ for $t \in [-\theta\sqrt{n}, \theta\sqrt{n}]$, where μ is a constant. By Lemma 8.16 (cf. (8.79)) and condition (I) we immediately see that $I_n'' = O(n^\beta e^{-\mu \log^2 n})$, which decays faster than any polynomial.

Now we estimate I_n'. Observe first that for $t \in [-\log n, \log n]$ we can apply the Taylor expansion to obtain

$$\begin{aligned}
h_n(t) &= \exp\left(n\left(e^{it/\sqrt{n}} - 1 - it/\sqrt{n}\right)\right) \\
&= e^{-t^2/2}\left(1 - \frac{it^3}{6\sqrt{n}} + \frac{t^4}{24n} - \frac{t^6}{72n} + O\left(\frac{\log^9 n}{n\sqrt{n}}\right)\right)
\end{aligned}$$

Furthermore, using (I) and Lemma 10.1 (with $D' = (1+\Omega)R$ and $z \in \mathcal{S}_{\theta'}$ for $\theta' < \theta$) we have $|\widetilde{G}'(z)| \le B_1|z|^{\beta-1}$ and $|\widetilde{G}''(z)| \le B_2|z|^{\beta-2}$, for some constants B_1 and B_2. Thus we can expand $\widetilde{G}(ne^{it/\sqrt{n}})$ around $t = 0$ as follows

$$\widetilde{G}(ne^{it/\sqrt{n}}) = \widetilde{G}(n) + it\sqrt{n}\widetilde{G}'(n) + t^2\Delta_n(t),$$

where

$$|\Delta_n(t)| \le (B_1 + B_2)n^{\beta-1}$$

since for t in the vicinity of zero

$$\Delta_n(t) = -\widetilde{G}'(ne^{it/\sqrt{n}})e^{it/\sqrt{n}} - \widetilde{G}''(e^{it/\sqrt{n}})ne^{2it/\sqrt{n}}.$$

In summary, the integral I'_n can be written as

$$\begin{aligned} I'_n &= \frac{1}{\sqrt{2\pi}}\int_{-\log n}^{\log n} e^{-t^2/2}\left(\widetilde{G}(n)+\widetilde{G}'(n)it\sqrt{n}\right)\left(1-\frac{it^3}{6\sqrt{n}}+\frac{t^4}{24n}-\frac{t^6}{72n}\right)dt \qquad (1.20)\\ &+ \frac{1}{\sqrt{2\pi}}\int_{-\log n}^{\log n} e^{-t^2/2}\Delta_n(t)t^2h_n(t)dt \qquad (1.21)\\ &+ \frac{1}{\sqrt{2\pi}}\int_{-\log n}^{\log n} e^{-t^2/2}\left(\widetilde{G}(n)+\widetilde{G}'(n)it\sqrt{n}\right)O\left(\frac{\log^9 n}{n\sqrt{n}}\right)dt. \qquad (1.22)\end{aligned}$$

To complete the proof we must estimate the above three integrals. From Lemma 8.16 and 10.1 we see that the first integral is equal to $\widetilde{G}(n)+O(n^{\beta-1})$. Using our estimate on $\Delta_n(t)$ we observe that the absolute value of the second integral is smaller than $(B_1+B_2)n^{\beta-1}$. Finally, the last integral is $O(n^{\beta-\frac{3}{2}}\log^9 n)$. All together the error is $O(n^{\beta-1})$ and Theorem 10.3 is proved. ■

Example 1.4 *Conditions (O) and (I) violated.*

Let $g_n = (-1)^n$, thus $\widetilde{G}(z) = e^{-2z}$ and condition (I) is true for any β and for any $\theta < \pi/2$. But in this case the condition (O) outside the cone $\mathcal{S}_\theta$ does not hold because $\widetilde{G}(z)e^z = e^{|z|}$ for $\arg(z) = \pi$. Clearly, g_n is not asymptotically equivalent to $\widetilde{G}(n)$.

If $g_n = (1+t)^n$ for $t > 0$, then $\widetilde{G}(z) = e^{tz}$. Condition (O) holds for some θ such that $(1+t)\cos\theta < 1$. But condition (I) inside the cone $\mathcal{S}_\theta$ does not hold because $\widetilde{G}(z)$ does not have a polynomial growth. And, as a matter of fact, g_n is not asymptotically equivalent to $\widetilde{G}(n)$. □

In some situations verifying condition (O) is quite troublesome. Fortunately, for some sequences having an analytic continuation of polynomial growth, condition (O) is automatically satisfied, as shown below.

Theorem 1.4 (Jacquet and Szpankowski, 1999) *Let $g(z)$ be an analytic continuation of $g(n) = g_n$ such that $g(z) = O(z^\beta)$ in a linear cone $\mathcal{S}_{\theta_0}$. Then for some θ_0 and for all linear cones $\mathcal{S}_\theta$ $(\theta < \theta_0)$, there exist $\alpha < 1$ and $A > 0$ such that*

$$z \notin \mathcal{S}_\theta \quad \Rightarrow \quad |\widetilde{G}(z)e^z| \le Ae^{\alpha|z|},$$

where $\widetilde{G}(z)$ is the Poisson transform of g_n.

Proof: Let $\mathcal{S}_{\theta_0}$ be the linear cone for which the polynomial bound over $g(z)$ holds. Let also $g^\star(s)$ be the Laplace transform of the function $g(x)$ of a real variable x, that is,

$$g^\star(s) = \int_0^\infty g(x)e^{-sx}dx$$

defined for $\Re(s) > 0$. It is well known (cf. [101]) that the inverse Laplace transform of $g^\star(s)$ exists in $\Re(s) > 0$ and one can write

$$g(x) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} g^\star(s)e^{sx}ds$$

with $\delta > 0$. In addition, $g^\star(s)$ is absolutely integrable on the line of integration. Observe now that the exponential generating function $G(z) = \widetilde{G}(z)e^z$ of $g(n)$ can be represented in terms of $g^\star(s)$ as follows

$$\begin{aligned} G(z) &= \sum_{n=0}^{\infty} g(n)\frac{z^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} g^\star(s)\exp(ns)\frac{z^n}{n!}ds \\ &= \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} g^\star(s)\sum_{n=0}^{\infty}\exp(ns)\frac{z^n}{n!}ds = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} g^\star(s)\exp\left(ze^s\right)ds, \end{aligned}$$

where the interchange of the integral and the summation is justified since both converge absolutely in their domains of definition. Thus the Poisson transform $\widetilde{G}(z)$ becomes

$$\widetilde{G}(z)e^z = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} g^\star(s)e^{e^s z}ds \tag{1.23}$$

for $\delta > 0$.

To take advantage of the above formula, we need an extension of the Laplace transform of a real variable to a complex variable, as we did in Section 9.3 for the Mellin transform. This fact is rather well known (e.g., [101, 217]), however, the reader is asked in Exercise 6 to derive such an extension. In particular, this exercise asks to show that if $g(z)$ is an analytic continuation of $g(x)$ in a cone $\mathcal{S}_\theta$, where $\theta < \theta_0$, then the inverse Laplace $g^\star(s)$ of the function $g(z)$ (of complex z) exists in a *bigger* cone $\mathcal{S}_{\theta+\pi/2}$ for all $\theta < \theta_0$ *provided* $g(z) = O(z^\beta)$ in the cone $\mathcal{S}_{\theta_0}$ (i.e., $g(z)$ is of polynomial growth).

In view of this we can write (10.23) as

$$\widetilde{G}(z)e^z = \frac{1}{2\pi i}\int_{\mathcal{L}_\varepsilon} g^\star(s)e^{e^s z}ds, \tag{1.24}$$

where $\mathcal{L}_\varepsilon$ is a piece-linear curve that parallels the boundary of the cone $\mathcal{S}_{\theta+\pi/2}$ at distance ε. That is, in $s = x + iy$ coordinates, the curve $\mathcal{L}_\varepsilon$ can be described as

$$y = \mathrm{sign}(y)\tan(\pi/2+\theta)(x-\varepsilon),$$

where $\mathrm{sign}(y)$ is the sign function (i.e., equal to 1 when $y \geq 0$ and -1 otherwise). We also define $\mathcal{L}_0$ as a curve obtained from $\mathcal{L}_\varepsilon$ as $\varepsilon \to 0$, that is, having the description $y = \mathrm{sign}(y)x\tan(\pi/2+\theta)$.

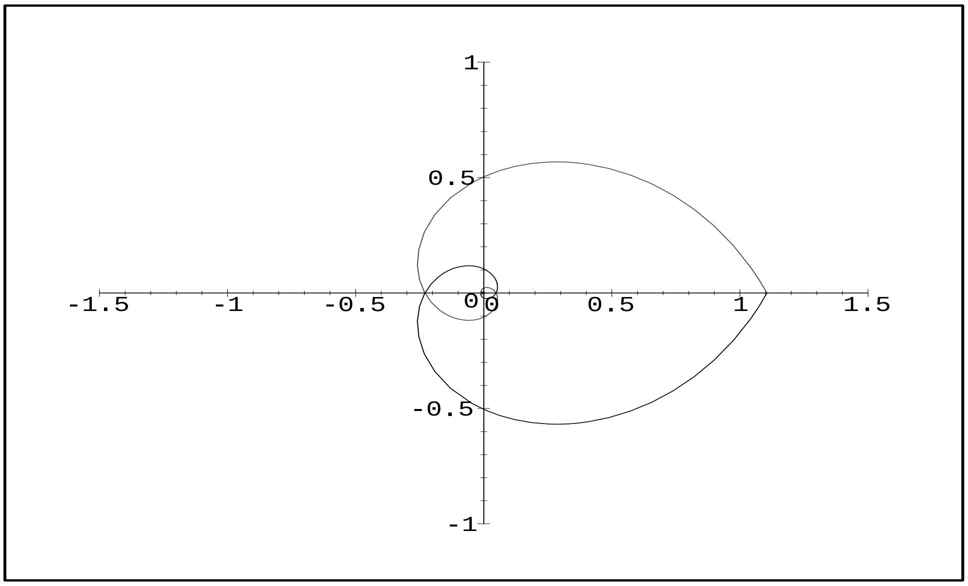

Figure 1.2: Image of $\mathcal{L}_\varepsilon$ with $\varepsilon = 0.1$ by e^s.

Using (10.24), we can now upper bound $\widetilde{G}(z)e^z$ as follows

$$|\widetilde{G}(z)e^z| \le \frac{1}{2\pi}\int_{\mathcal{L}_\varepsilon} |g^\star(s)| \exp(\Re(e^s z))ds \le \frac{1}{2\pi}\int_{\mathcal{L}_\varepsilon} |g^\star(s)| \exp\left(\Re(e^s e^{i\theta})|z|\right) ds, \qquad (1.25)$$

since for $z \notin \mathcal{S}_\theta$ we have $\cos(\arg z) \le \cos\theta$ for $|\theta| \le \pi/2$. To complete the proof, we must show that $\Re(e^s e^{i\theta}) < \alpha$ for some $\alpha < 1$ and all $s \in \mathcal{L}_\varepsilon$. If this is true, we obtain immediately $|\widetilde{G}(z)e^z| \le Ae^{\alpha|z|}$ for $\alpha < 1$, where $A = \frac{1}{2\pi}\int_{\mathcal{L}_\varepsilon} |g^\star(s)|ds < \infty$.

We concentrate now on showing that $\Re(e^s e^{i\theta}) < \alpha < 1$. We study the image $\mathcal{I}_\varepsilon$ of $\mathcal{L}_\varepsilon$ under the function e^s which is plotted in Figure 10.2. (In fact, in Figure 10.2 we assume $\theta = \pi/6$, and $\mathcal{I}_\varepsilon$ with $\varepsilon = 0.1$ has the following parametric description: $(\exp(-t/2 + 0.1)\cos(\pm t),\ \exp(-t/2+0.1)\sin(\pm t))$.) When $\varepsilon \to 0$, this image tends to the image $\mathcal{I}_0$ of $\mathcal{L}_0$. Observe that $\mathcal{I}_0$ is contained in the unit disk, and $\mathcal{I}_0$ touches the unit circle only at $s = 0$. We rewrite (10.24) as

$$\widetilde{G}(z)e^z = \frac{1}{2\pi i}\int_{\mathcal{L}_\varepsilon} g^\star(s)\exp[e^s e^{i\arg(z)}|z|]ds$$

to see that the image $\mathcal{I}_\varepsilon$ of $\mathcal{L}_\varepsilon$ is rotated by the angle $\arg(z)$. Let us now fix an arbitrary $0 < \theta < \theta_0$, and consider for a moment only the image $\mathcal{I}_0$ of $\mathcal{L}_0$. Observe that if one rotates the image $\mathcal{I}_0$ by a non zero argument, then the new image has the *real part* strictly less than 1. In fact, the real part will be smaller than $1 - O(\theta^2)$. Finally, considering the image $\mathcal{I}_\varepsilon$ of $\mathcal{L}_\varepsilon$, we can easily choose ε so that the real part of the rotated images of $\mathcal{L}_\varepsilon$ remains

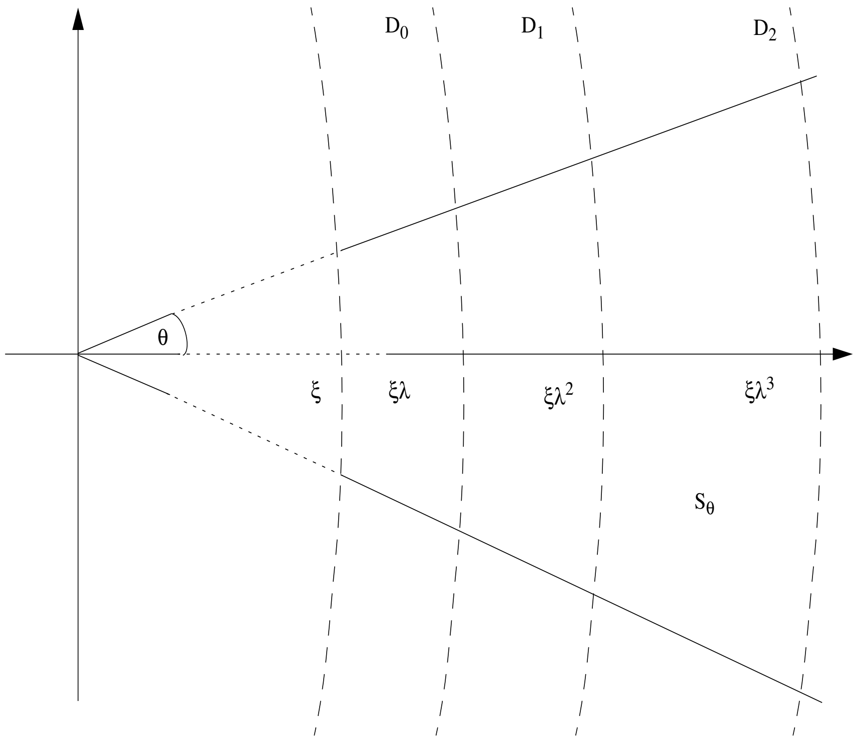

Figure 1.3: The linear cone $\mathcal{S}_\theta$ and the increasing domains $\mathcal{D}_m$.

smaller than some α such that $1 - O(\theta^2) < \alpha < 1$ for all arguments greater than θ. Thus, $\Re(e^s e^{i\theta}) < \alpha < 1$, which completes the proof. ∎

Before we illustrate the above theorem (cf. Example 7 below), we provide another approach to establish condition (O) that is useful when dealing with Poisson transforms expressed as linear functional equations. In this case conditions (I) and (O) can be verified relatively easily by applying mathematical induction over the so–called **increasing domains** that we define next.

Lee us consider the following general functional equation

$$\widetilde{G}(z) = \gamma_1(z)\widetilde{G}(zp) + \gamma_2(z)\widetilde{G}(zq) + t(z), \tag{1.26}$$

where $\gamma_1(z)$, $\gamma_2(z)$ and $t(z)$ are functions of z such that the above equation has a solution. The reader is referred to Fayolle *et al.* [117] (cf. also [375]) for conditions on these functions under which a solution of (10.26) exists. We further assume that $p + q = 1$, but in fact zp and zq could be replaced by more general functions that form a semi group of substitutions under the operation of composition of functions. We recall that we analyzed a simple version of (10.26) in Section 7.6.1.

Let us define, for integers $m = 0, 1, \ldots,$ and a constant λ such that $0 < \max\{p, q\} \leq \lambda^{-1} < 1$, a sequence of increasing domains (cf. Figure 10.3) $\mathcal{D}_m$ as

$$\mathcal{D}_m = \{z : \ \xi\delta \leq |z| \leq \xi\lambda^{m+1}\}$$

for some constant $\xi > 0$ and $\delta \leq \min\{p, q\}$. Observe that

$$z \in \mathcal{D}_{m+1} - D_m \qquad \Rightarrow \qquad pz, \ qz \in \mathcal{D}_m \ . \tag{1.27}$$

The last property is crucial for applying mathematical induction over m in order to establish appropriate bounds on $\widetilde{G}(z)$ over the whole complex plane.

Theorem 1.5 *Consider the following functional equation (with $p + q = 1$)*

$$\widetilde{G}(z) = \gamma_1(z)\widetilde{G}(zp) + \gamma_2(z)\widetilde{G}(zq) + t(z)$$

that is postulated to have an entire solution. Let for some positive β, $0 < \theta < \pi/2$ and $0 < \eta < 1$ the following conditions hold for $|z| > \xi$:

(I) For $z \in \mathcal{S}_\theta$

$$\begin{aligned} |\gamma_1(z)|p^\beta + |\gamma_2(z)|q^\beta &\leq 1 - \eta, & (1.28)\\ |t(z)| &\leq B\eta|z|^\beta \ ; & (1.29) \end{aligned}$$

(O) For $z \notin \mathcal{S}_\theta$ and for some $\alpha < 1$ the following three inequalities are true

$$\begin{aligned} |\gamma_1(z)|e^{q\Re(z)} &\leq \frac{1}{3}e^{\alpha|z|q}, & (1.30)\\ |\gamma_2(z)|e^{p\Re(z)} &\leq \frac{1}{3}e^{\alpha|z|p}, & (1.31)\\ |t(z)|e^{\Re(z)} &\leq \frac{1}{3}e^{\alpha|z|} \ . & (1.32) \end{aligned}$$

Then

$$g_n = \widetilde{G}(n) + O(n^{\beta-1}), \tag{1.33}$$

where $g_n = [z^n]\left(n!e^z\widetilde{G}(z)\right)$.

Proof. We apply Theorem 10.3. For this, we need to establish the following bounds

$$|\widetilde{G}(z)| \leq B|z|^\beta \tag{1.34}$$

for $z \in \mathcal{S}_\theta$, and

$$|\widetilde{G}(z)e^z| \leq B'e^{\alpha|z|} \tag{1.35}$$

for $z \notin \mathcal{S}_\theta$ with $\alpha < 1$, where $B, B' > 0$ are constants. It suffices to prove that under (10.28), (10.29) and (10.30)– (10.32) the above two conditions hold. We start with the proof of (10.34). We apply induction over the increasing domains $\mathcal{D}_m$ (cf. Figure 10.3). Let us first consider $z \in \mathcal{S}_\theta$. Define $\hat{\mathcal{D}}_m = \mathcal{D}_m \cap \mathcal{S}_\theta$. We take B large enough such that for $m = 0$ the above inequalities hold for $|z| \le \xi$. Let us assume now that inequality (10.34) is satisfied in $\hat{\mathcal{D}}_m$, and we prove that it also holds in a larger region, namely $\hat{\mathcal{D}}_{m+1}$, thus proving (10.34) in $\mathcal{S}_\theta$. For this it suffices to consider $z \in \hat{\mathcal{D}}_{m+1} - \hat{\mathcal{D}}_m$. By (10.27) we have $zp, zq \in \hat{\mathcal{D}}_m$, and we can invoke our induction hypothesis. Hence, taking into account (10.28) and (10.29) we conclude from equation (10.26) that

$$\begin{aligned}
|\widetilde{G}(z)| &\le |\gamma_1(z)||\widetilde{G}(zp)| + |\gamma_2(z)||\widetilde{G}(zq)| + |t(z)| \ , \\
&\le B|\gamma_1(z)||z|^\beta p^\beta + B|\gamma_2(z)||z|^\beta q^\beta + |t(z)| \ , \\
&\le B(1-\eta)|z|^\beta + B\eta|z|^\beta = B|z|^\beta
\end{aligned}$$

which is the desired bound.

Assume now that $z \notin \mathcal{S}_\theta$ and we aim at proving (10.35). We first observe that $|e^z| = e^{\Re(z)} \le e^{\alpha|z|}$, where $\alpha \ge \cos\theta \ge \cos(\arg(z))$ The induction over the increasing domains can be applied as before, however, this time we consider $\overline{\mathcal{D}}_m = \mathcal{D}_m \cap \overline{\mathcal{S}}_\theta$, where $\overline{\mathcal{S}}_\theta$ is the complementary set to $\mathcal{S}_\theta$. Observe that

$$\begin{aligned}
|\widetilde{G}(z)e^z| &\le |\gamma_1(z)||\widetilde{G}(zp)e^{zp}||e^{zq}| + |\gamma_2(z)||\widetilde{G}(zq)e^{zq}||e^{zp}| + |t(z)e^z| \ , \\
&\le B'|\gamma_1(z)|e^{\alpha|z|p}e^{q\Re(z)} + B'|\gamma_2(z)|e^{\alpha|z|q}e^{p\Re(z)} + |t(z)|e^{\Re(z)} \ , \\
&\le B'e^{\alpha|z|},
\end{aligned}$$

where the last inequality follows from (10.30)-(10.32) and $\Re(z) = |z|\cos(\arg(z)) \le |z|\cos\theta \le \alpha|z|$. This completes the proof. ∎

Example 1.5 *Conflict Resolution Algorithm*

We consider here a conflict resolution algorithm. While its description can be found in Capetenakis [63], and Tsybakov and Mikhailiov [423], we provide here a brief review of the *interval searching algorithm*. The access to a slotted broadcast channel is controlled by a window based mechanism that works as follows: This window will be referred to as the *enabled interval* (EI). Let S_i denote the starting point for the i-th EI, and t_i is the corresponding starting point for the *conflict resolution interval* (CRI), where CRI represents the number of slots needed to resolve a collision. Roughly speaking, at each step of the algorithm, we compute the endpoints of the EI based on the outcome of the channel. The parameters of interest are: the length of the conflict resolution interval T_n, the fraction of resolved interval W_n, and the number of resolved packets C_n. For example, the Poisson transform $\widetilde{C}(z)$ of C_n satisfies

$$\widetilde{C}(z) = \left(1 + (1 + z/2)e^{-z/2}\right)\widetilde{C}(z/2) \ .$$

Indeed, if there is at most one packet in the first half of EI – which happens with probability $(1+z/2)e^{-z/2}$ in the Poisson model – then we explore the second half of the interval represented by $\widetilde{C}(z/2)$. Otherwise, we explore the first half of the interval, and thus the first term in the above equation. This equation is of the form of (10.26), thus falls under Theorem 10.5. Jacquet and Szpankowski [212] proved that for large $z \in \mathcal{S}_\theta$ the above equation admits asymptotically the following solution (cf. [212])

$$\widetilde{C}(z) = D + P(\log z) + O(1/z), \tag{1.36}$$

where

$$D = \exp\left(\frac{1}{\log 2}\int_0^\infty \frac{xe^{-x}\log x}{1+(1+x)e^{-x}}dx + \frac{1}{2}\log 2\right) \approx 2.505,$$

and $P(\log z)$ is our familiar fluctuating function with a small amplitude. This can be established by the method discussed in Chapter 9 and the reader is asked to prove it in Exercise 5. To depoissonize the solution, one must check conditions (I) and (O) of Theorem 10.3. But $\widetilde{C}(z) = O(1)$ in a cone $\mathcal{S}_\theta$, thus we need only to verify (O) outside the cone $\mathcal{S}_\theta$. By Theorem 10.5 (e.g., see (10.30)) we need to show the existence of ξ such that for $|z| > \xi$ the following holds

$$(1+(1+|z|/2))e^{\Re(z)/2} \le e^{\alpha|z|/2}$$

which is clearly true for large enough ξ since $\alpha \ge \cos\theta$. By Theorem 10.5 we conclude that $C_n = D + P(\log n) + O(n^{-1})$. □

1.3 Generalizations of Depoissonization

In this section we generalize our basic depoissonization Theorem 10.3 twofold: (i) we derive a full asymptotic expansion for g_n, and (ii) we enlarge the class of Poisson transforms $\widetilde{G}(z)$ for which the depoissonization holds (i.e., we allow for exponential rather than algebraic bounds on $\widetilde{G}(z)$). We prove all of these results in a uniform manner in Section 10.3.3 using a general depoissonization tool presented in Theorem 10.13.

1.3.1 A Full Asymptotic Depoissonization

Before we present a precise statement, let us derive formally a full asymptotic expansion of g_n. We shall seek such an expansion in the following form

$$g_n = \sum_{i,j\ge 0} b_{ij} n^i \widetilde{G}^{(j)}(n).$$

To identify formally the coefficients b_{ij} we proceed as follows. By Taylor series

$$\widetilde{G}(z) = \sum_{j\ge 0} \frac{(z-n)^j}{j!}\widetilde{G}^{(j)}(n) \ ,$$

which yields

$$g_n = n![z^n](e^z\widetilde{G}(z)) = \sum_j n![z^n]\frac{(z-n)^j e^z}{j!}\widetilde{G}^{(j)}(n).$$

Hence, we formally set

$$b_{ij} = [n^i][z^n]\frac{n!(z-n)^j}{j!}e^z.$$

After interchanging indexes i and j, factorizing the operator $[z^n]$, noticing that $\sum_i n^i[n^i]f(n,z) = f(n,z)$ for any analytic function $f(n,z)$, and setting formally $y^j = \widetilde{G}^{(j)}(z)$ we obtain

$$\begin{aligned}
\sum_{ij} b_{ij}n^i y^j &= [z^n]\sum_j y^j \sum_i n^i[n^i]\frac{n!(z-n)^j e^z}{j!} \\
&= [z^n]\sum_j \frac{n!(z-n)^j y^j e^z}{j!} = [z^n]n!\exp((z-n)y+z) \\
&= (1+y)^n e^{-ny} = \exp(n\ln(1+y)-ny).
\end{aligned}$$

Thus

$$b_{ij} = [x^i][y^j]\exp(n\ln(1+y)-ny).$$

We observe that $b_{ij}=0$ for $j<2i$. Indeed, as in the case of the coefficients a_{ij} discussed in Section 10.1, we let $f(x,y)=\exp(x\log(1+y)-xy)$ and observe that $f(xy^{-2},y)$ is analytic at $x=y=0$. Hence, its Laurent expansion possesses only terms like $x^i y^{j-2i}$ with nonnegative exponents. This means that for $j<2i$ the coefficients $b_{ij}=0$.

We are now ready to formulate a general depoissonization result with polynomial bounds. We shall defer the proof until Section 10.3.3, where we present a general depoissonization tool that allows us to establish all results of this section in a uniform manner.

Theorem 1.6 (Jacquet and Szpankowski, 1998) *Let the following two conditions hold for some numbers A, B, $R>0$ and $\alpha>0$, β, and γ:*
(I) *For $z\in\mathcal{S}_\theta$, $0<|\theta|<\pi/2$,*

$$|z|>R \quad\Rightarrow\quad |\widetilde{G}(z)| \le B|z|^\beta\Psi(|z|), \tag{1.37}$$

where $\Psi(x)$ is a slowly varying function , that is, such that for fixed t $\lim_{x\to\infty}\frac{\Psi(tx)}{\Psi(x)}=1$ (e.g., $\Psi(x)=\log^d x$ for some $d>0$);
(O) *For all $z=\rho e^{i\vartheta}$ such that $z\notin\mathcal{S}_\theta$ we have*

$$\rho=|z|>R \quad\Rightarrow\quad |\widetilde{G}(z)e^z| \le A\rho^\gamma\exp[(1-\alpha\vartheta^2)\rho]. \tag{1.38}$$

Then for every nonnegative integer m

$$\begin{aligned} g_n &= \sum_{i=0}^{m}\sum_{j=0}^{i+m} b_{ij} n^i \widetilde{G}^{(j)}(n) + O(n^{\beta-m-1}\Psi(n)) \qquad (1.39)\\ &= \widetilde{G}(n) + \sum_{k=1}^{m}\sum_{i=1}^{k} b_{i,k+i} n^i \widetilde{G}^{(k+i)}(n) + O(n^{\beta-m-1}\Psi(n)), \end{aligned}$$

where b_{ij} *are as the coefficients of* $\exp(x\log(1+y)-xy)$ *at* $x^i y^j$, *that is,*

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b_{ij} x^i y^j = \exp(x\log(1+y)-xy)\ . \qquad (1.40)$$

with $b_{ij}=0$ *for* $j<2i$.

The first few terms of the above expansion are

$$\begin{aligned} g_n = \ & \widetilde{G}(n) - \frac{1}{2}n\widetilde{G}^{(2)}(n) + \frac{1}{3}n\widetilde{G}^{(3)}(n) + \frac{1}{8}n^2\widetilde{G}^{(4)}(n) - \frac{1}{4}n\widetilde{G}^{(4)}(n) - \frac{1}{6}n^2\widetilde{G}^{(5)}(n) - \\ & -\frac{1}{48}n^3\widetilde{G}^{(6)}(n) + \frac{1}{5}n\widetilde{G}^{(5)}(n) + \frac{13}{72}n^2\widetilde{G}^{(6)}(n) + \frac{1}{24}n^3\widetilde{G}^{(7)}(n) + \frac{1}{384}n^4\widetilde{G}^{(8)}(n) - \\ & -\frac{1}{6}n\widetilde{G}^{(6)}(n) - \frac{11}{60}n^2\widetilde{G}^{(7)}(n) - \frac{17}{288}n^3\widetilde{G}^{(8)}(n) - \frac{1}{144}n^4\widetilde{G}^{(9)}(n) - \frac{1}{3840}n^5\widetilde{G}^{(10)}(n) \\ & +O(n^{\beta-6})\ . \end{aligned}$$

Example 1.6 *Flajolet's Dirichlet Depoissonization*
Let

$$g_n = \sum_{k=0}^{\infty}\left(1-(1-2^{-k})^n\right) \qquad (1.41)$$

whose Poisson transform is (cf. Item 7 of Table 10.1)

$$\widetilde{G}(z) = \sum_{k=0}^{\infty}\left(1-e^{-z2^{-k}}\right),$$

and the jth derivative is

$$\widetilde{G}^{(j)}(z) = (-1)^{j+1}\sum_{k=0}^{\infty} 2^{-jk} e^{-z2^{-k}}.$$

We are interested in a full asymptotic expansion of g_n using the depoissonization Theorem 10.6. For this we need to find the asymptotics of $\widetilde{G}^{(j)}(z)$ for z in a cone $\mathcal{S}_\theta$. But $\widetilde{G}^{(j)}(z)$

is a harmonic sum, hence by Theorem 9.1 the Mellin transforms are

$$\begin{aligned}\mathcal{M}[\widetilde{G}(z);s] &= -\frac{\Gamma(s)}{1-2^s}, \qquad -1<\Re(s)<0,\\ \mathcal{M}[\widetilde{G}^{(j)}(z);s] &= \frac{(-1)^{j+1}\Gamma(s)}{1-2^{s-j}}, \qquad 0<\Re(s)<j, \quad j\geq 1.\end{aligned}$$

Using the reverse mapping Theorem 9.2 we find for $x\to\infty$ (which by Theorem 9.5 can be extended to complex $z\to\infty$ in a cone $\mathcal{S}_\theta$ with $|\theta|<\pi/2$)

$$\begin{aligned}\widetilde{G}(x) &= \log_2 x + \frac{\gamma}{\log 2} + \frac{1}{2} + P_0(\log_2 x) + O(x^{-M}),\\ \widetilde{G}^{(j)}(x) &= \frac{(-1)^{j+1}}{x^j}P_j(\log_2 x) + O(x^{-M}), \quad j\geq 1\end{aligned}$$

for any $M>0$, where

$$\begin{aligned}P_0(\log_2 x) &= \frac{1}{\log 2}\sum_{\ell\neq 0}\Gamma(2\pi i\ell/\log 2)e^{-2\pi i\ell\log_2 x}\\ P_j(\log_2 x) &= \frac{1}{\log 2}\sum_{\ell=-\infty}^{\infty}\Gamma(j+2\pi i\ell/\log 2)e^{-2\pi i\ell\log_2 x}, \qquad j\geq 1.\end{aligned}$$

Since $g(z)=O(\log z)$ inside a cone $\mathcal{S}_\theta$, hence by Theorem 10.4 and Theorem 10.6 we finally obtain

$$g_n = \log_2 n + \frac{\gamma}{\log 2} + \frac{1}{2} + P_0(\log_2 n) + \sum_{k=1}^{m}\sum_{i=1}^{k}(-1)^{k+i+1}b_{i,k+i}n^{-k}P_{k+i}(\log_2 n) + O(n^{-m-1}\log n) \tag{1.42}$$

for any $m\geq 1$, where b_{ij} are given by (10.40).

Flajolet [126] suggested to consider a more general sum, namely

$$g_n = \sum_{k\geq 0} a_k(1-(1-a_k)^n)$$

for a sequence $a_k\to 0$. From Item 7 of Table 10.1 we know that its Poisson transform is

$$\widetilde{G}(z) = \sum_{k\geq 0} a_k(1-e^{-za_k}),$$

thus one can use depoissonization, as we did above, to find the full asymptotic expansion of g_n. Flajolet (cf. [66]) proposed another approach that he calls the **Dirichlet depoissonization**. It is based on the following observation. Define the following two Dirichlet series

$$\Lambda(s) = \sum_{k\geq 0} a_k^s, \qquad \Omega(s) = \sum_{k\geq 0}\left(\log\frac{1}{1-a_k}\right)^s.$$

Formally, we have

$$\begin{aligned}
\Omega(s) &= \sum_{k\geq 0} \exp\left(s \log\log(1-a_k)^{-1}\right) = \sum_{k\geq 0} \exp\left(s \log\left(a_k + \frac{1}{2}a_k^2 + \frac{1}{3}a_k^3 + \cdots\right)\right) \\
&= \sum_{k\geq 0} a_k^s \exp\left(s \log\left(1 + \frac{1}{2}a_k + \frac{1}{3}a_k^2 + \cdots\right)\right) \\
&= \sum_{k\geq 0} a_k^s \exp\left(s \left(\frac{1}{2}a_k + \frac{5}{24}a_k^2 + \cdots\right)\right) \\
&= \sum_{k\geq 0} a_k^s \left(1 + \frac{1}{2}sa_k + (\frac{5}{24}s + \frac{1}{8}s^2)a_k^2 + (\frac{1}{8}s + \frac{5}{48}s^2 + \frac{1}{48}s^3)a_k^3 + \cdots\right) \\
&= \Lambda(s) + c_1(s)\Lambda(s+1) + c_2(s)\Lambda(s+2) + \cdots .
\end{aligned}$$

Thus the singularities of $\Omega(s)$ are those of $\Lambda(s)$ plus $\{0, -1, \ldots\}$. This was rigorously proved in [66] from where we quote the following result.

Lemma 1.7 (Dirichlet Depoissonization) *Let $a_k \to 0$ with a_k positive. Define the above two Dirichlet series $\Lambda(s)$ and $\Omega(s)$. Then, if the Dirichlet series $\Lambda(s)$ of a_k is meromorphic in the whole complex plane and it is of "weak polynomial growth" (i.e., the growth is only controlled on certain lines parallel to the real axis tending to $\pm i\infty$), then the same properties hold for $\Omega(s)$. In addition, the singularities of Λ, Ω are related by*

$$\mathrm{Sing}(\Omega) = \mathrm{Sing}(\Lambda) + \{0, -1, -2, \cdots\},$$

and the singular expansions of $\Omega(s)$ and $\Lambda(s)$ are related by

$$\Omega(s) \asymp \Lambda(s) + c_1(s)\Lambda(s+1) + c_2(s)\Lambda(s+2) + \cdots, \tag{1.43}$$

where

$$c_j(s) = [x^j] \exp\left(\frac{s}{x} \log \frac{1}{1-x}\right). \tag{1.44}$$

For g_n defined in (10.41) we have

$$\Omega(-s) \asymp \Lambda(-s) + \frac{s}{2}\Lambda(-s+1) + (\frac{5s}{24} + \frac{s^2}{8})\Lambda(-s+2) + \cdots ,$$

thus (10.42) is recovered.

Example 1.7 *Entropy of Binomial Distribution*

We compute here a full asymptotic expansion of Shannon entropy h_n of the binomial distribution $\binom{n}{k}p^k q^{n-k}$ $(q = 1-p)$, that is,

$$h_n = -\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \log\left(\binom{n}{k} p^k q^{n-k}\right).$$

We shall follow the approach suggested in Jacquet and Szpankowski [217], while the reader is asked in Exercise 10 to use the singularity approach to obtain the same result (cf. Flajolet [127]). Observe that

$$\begin{aligned} h_n &= -\ln(n!) - n(p\ln p + q\ln q) \\ &+ \sum_{k=0}^{n} \ln(k!)\binom{n}{k}p^k q^{n-k} + \sum_{k=0}^{n}\ln((n-k)!)\binom{n}{k}p^k q^{n-k} . \end{aligned}$$

We use analytic poissonization and depoissonization to estimate

$$g_n := \sum_{k=0}^{n} \ln(k!)\binom{n}{k}p^k q^{n-k} + \sum_{k=0}^{n}\ln((n-k)!)\binom{n}{k}p^k q^{n-k}.$$

Observe that the Poisson transform $\widetilde{G}(z)$ of g_n is

$$\widetilde{G}(z) = \widetilde{F}(zp) + \widetilde{F}(zq),$$

where

$$\widetilde{F}(z) = \sum_{n=0}^{\infty} \ln(n!)\frac{z^n}{n!}e^{-z}.$$

Thus

$$h_n = -\ln(n!) - n(p\ln p + q\ln q) + n![z^n]\left(e^z\widetilde{F}(zp) + e^z\widetilde{F}(zq)\right).$$

We cannot apply directly the depoissonization Theorem 10.6 since the Poisson transform $\widetilde{F}(z)$ is not given in a closed-form. But, we can apply our previous analytic poissonization Theorem 10.2 to find $\widetilde{F}(z)$ for $z \to \infty$ and then use Theorem 10.6. In fact, in Example 2 we already computed $\widetilde{F}(z)$ asymptotically. Setting $\log(n!) = f_n = f(n) = \log\Gamma(n+1)$ and using the symbolic notation proposed in Section 10.2 we can formally write for the k-th derivative of $\widetilde{F}(z)$

$$\widetilde{F}^{(k)}(\lambda n) = (e^y - 1)^k \exp\left(n\lambda(e^y-1) - n\lambda y\right), \qquad y^j = f^{(j)}(\lambda n),$$

where λ is equal to either p or q. Since f_n has an analytic continuation of a polynomial growth, we can apply Theorem 10.6 to obtain

$$\begin{aligned} n![z^n]\left(e^z\widetilde{F}(\lambda z)\right) &= \sum_{i,j\geq 0} b_{ij}n^i\lambda^j(e^y-1)^j \exp\left(n\lambda(e^y-1) - n\lambda y\right), \quad && y^j = f^{(j)}(\lambda n) \\ &= \exp\left(n\log\left(1+\lambda(e^y-1)\right) - \lambda n y\right), && y^j = f^{(j)}(\lambda n), \end{aligned}$$

where we used (10.40) in the last line. Introducing $c_{i,k}(\lambda)$ as follows

$$\exp\left(x\log\left(1+\lambda(e^y-1)\right) - \lambda x y\right) = \sum_{i=0}^{\infty}\sum_{k=2i}^{\infty} c_{i,k}(\lambda)x^i y^k,$$

we finally arrive at

$$\begin{aligned} h_n &= -f(n) + f(np) + f(nq) - n(p\ln p + q\ln q) \\ &+ \sum_{i=1}^{\infty}\sum_{k=2i}^{\infty} n^i \left(c_{i,k}(p)p^i f^{(k)}(np) + c_{i,k}(q)q^i f^{(k)}(nq)\right), \end{aligned}$$

where $f^{(k)}(x)$ $(k \geq 1)$ are computed in Example 2. This leads to

$$h_n \sim \frac{1}{2}\ln n + \frac{1}{2} + \frac{1}{2}\ln(2\pi pq) + \sum_{m=1}^{\infty}\frac{e_m}{n^m}$$

and e_m can be explicitly computed, as shown in [217]. □

1.3.2 Exponential Depoissonization

In some applications the assumption about the polynomial growth of $\widetilde{G}(z)$ is too restrictive (cf. Example 3 of Chapter 9). Here we extend Theorem 10.6 to Poisson transforms with exponential growth $\exp(z^\beta)$ for $0 < \beta < \frac{1}{2}$. The proof is again delayed until the last subsection. However, below we provide a sketch of the proof for the basic exponential depoissonization.

Theorem 1.8 *Let the conditions of Theorem 10.6 be satisfied with condition (I) replaced by*

$$|\widetilde{G}(z)| \leq A\exp(B|z|^\beta) \tag{1.45}$$

for some $0 < \beta < \frac{1}{2}$ and constants $A > 0$ and B. Then for every integer $m \geq 0$

$$g_n = \sum_{i=0}^{m}\sum_{j=0}^{i+m} b_{ij}n^i\widetilde{G}^{(j)}(n) + O(n^{-(m+1)(1-2\beta)}\exp(Bn^\beta)) \tag{1.46}$$

for large n.

Sketch of Proof. It seems appropriate to give here a sketch of the proof for the leading term (cf. Section 10.3.3), namely,

$$g_n = \widetilde{G}(n) + O\left(n^{2\beta-1}\exp(Bn^\beta)\right). \tag{1.47}$$

This can be derived along the same lines as the proof of the basic depoissonization Theorem 10.3. As in the proof of Theorem 10.3, we can easily estimate the integrals E_n and I''_n, and therefore we are left with the integral I'_n (cf. (10.20)–(10.22)). To estimate I'_n we need an extension of Lemma 10.1 for functions $\widetilde{G}(z)$ with an exponential bound. We shall show that if (10.45) holds, then the kth derivative $\widetilde{G}^{(k)}(z)$ can be bounded as below

$$|\widetilde{G}^{(k)}(z)| \leq B_k|z|^{k(\beta-1)}\exp(B|z|^\beta). \tag{1.48}$$

Indeed, using Cauchy's bound (10.10) with $r = \Omega|z|^{1-\beta}$ we obtain

$$|\widetilde{G}^{(k)}(z)| \leq \frac{\max_{|\omega|=\Omega|z|^{1-\beta}} |\widetilde{G}(z+\omega)|}{\Omega|z|^{k(1-\beta)}},$$

where $z+\omega \in \mathcal{S}_\theta$, so that the exponential bound $|\widetilde{G}(z+\omega)| < A\exp(B|z+\omega|^\beta$ is still valid. Since the derivative of z^β is $O(z^{\beta-1})$ we have

$$\begin{aligned}\max_{|\omega|=\Omega|z|^{1-\beta}} |\widetilde{G}(z+\omega)| &\leq A\exp(|B||z|^\beta + O(z^{\beta-1})|z|^{1-\beta}\Omega) \\ &\leq A'\exp(|B||z|^\beta)\end{aligned}$$

for some constant A'. This proves (10.48).

Now we can consider the integral I'_n that we repeat below

$$\begin{aligned} I'_n &= \frac{1}{\sqrt{2\pi}}\int_{-\log n}^{\log n} e^{-t^2/2}\left(\widetilde{G}(n) + \widetilde{G}'(n)it\sqrt{n}\right)\left(1 - \frac{it^3}{6\sqrt{n}} + \frac{t^4}{24n} - \frac{t^6}{72n}\right)dt \\ &+ \frac{1}{\sqrt{2\pi}}\int_{-\log n}^{\log n} e^{-t^2/2}\Delta_n(t)t^2h_n(t)dt \\ &+ \frac{1}{\sqrt{2\pi}}\int_{-\log n}^{\log n} e^{-t^2/2}\left(\widetilde{G}(n) + \widetilde{G}'(n)it\sqrt{n}\right)O\left(\frac{\log^9 n}{n\sqrt{n}}\right)dt, \end{aligned}$$

where we recall that

$$\begin{aligned} h_n(t) &= \exp(n(e^{it/\sqrt{n}} - 1 - it/\sqrt{n})), \\ \Delta_n(t) &= -\widetilde{G}'(ne^{it/\sqrt{n}})e^{it/\sqrt{n}} - \widetilde{G}''(e^{it/\sqrt{n}})ne^{2it/\sqrt{n}}. \end{aligned}$$

Then (10.48) and the above yield $|\Delta_n(t)| = O(n^{2\beta-1}\exp(Bn^\beta))$. But the first part of the above integral is $\widetilde{G}(n) + O(n^{\beta-1}\exp(Bn^\beta))$ for $\beta > 0$, the second part is $O(n^{2\beta-1}\exp(Bn^\beta))$, and the last part is $O(n^{-3/2}\exp(Bn^\beta))$. This proves (10.47). ■

Example 1.8 *Example 3 of Chapter 9 Revisited*

In Example 3 of Chapter 9 we studied the sequence C_n satisfying the recurrence (9.31), that is,

$$C_n = 4\sum_{k=2}^{n}\binom{n}{k}\frac{2^{-n}}{n-k+1}C_k, \qquad n \geq 2$$

with $C_2 = 1$. We proved that its Poisson transform $\widetilde{C}(z)$ satisfies the following functional equation

$$\widetilde{C}(z) = \frac{8}{z}(1 - e^{-z/2})\widetilde{C}\left(\frac{z}{2}\right),$$

while $F(z) = \log z^2 \widetilde{C}(z)/2$ fulfills

$$F(2z) - F(z) = \log\left(\frac{1-e^{-z}}{z}\right). \tag{1.49}$$

In (9.32) of Example 3 in Chapter 9 we proved that

$$\exp[F(z)] \sim \sqrt{z}2^{-1/12}\exp\left(\frac{\gamma(1)+\gamma^2/2-\pi^2/12}{\log 2}\right)\exp\left[-\frac{1}{2}\frac{\log^2(z)}{\log 2}+\Psi(\log_2 z)\right],$$

where

$$\Psi(\log_2 z) = \sum_{\substack{\ell=-\infty\\ \ell\neq 0}}^{\infty}\frac{1}{2\pi i\ell}\Gamma\left(1-\frac{2\pi i\ell}{\log 2}\right)\zeta\left(1-\frac{2\pi i\ell}{\log 2}\right)e^{2\pi i\ell\log_2 z}.$$

Thus $\widetilde{C}(z) = O(z^{5/2}\exp(\log^2 z))$ in the cone $\mathcal{S}_\theta$ (by analytic continuation). By Theorem 10.8 we find

$$C_n \sim \frac{1}{2}n^{5/2}2^{-1/12}\exp\left(\frac{\gamma(1)+\gamma^2/2-\pi^2/12}{\log 2}\right)\exp\left[-\frac{1}{2}\frac{\log^2(j)}{\log 2}+\Psi(\log_2 n)\right]$$

as $n\to\infty$. □

1.3.3 General Depoissonization Tool

Here we prove all the depoissonization results discussed so far. We start with a few lemmas followed by our main depoissonization tool Theorem 10.13.

It turns out that in the course of the proof we often have to deal with functions satisfying certain property. We formalize it in the following definition.

Definition 1.9 *We say that a sequence of functions $f_n : I_n \to \mathbb{C}$ defined on subintervals I_n belongs to the class $\mathbb{D}_k(\omega)$ for ω real if there exist $D>0$ such that for all integers $j\le k$ and $x\in I_n$ we have $|f_n^{(j)}(x)| \le Dn^{-j\omega}$.*

Example 1.9 *Functions belonging to $\mathbb{D}_k(\omega)$.*

Consider the following two functions:

1. Let f be infinitely differentiable on $[-1,1]$, and define $f_n(x) = f(xn^{-\omega})$ on $I_n = [-n^\omega, n^\omega]$. Clearly, $f_n\in\mathbb{D}_k(\omega)$ for every $k\ge 0$.

2. Let F_n be an analytic function defined in $\{z : |z|\le n^\omega\}$, $n\ge 1$, such that the sequence F_n is uniformly bounded. Then the restriction f_n of F_n to $I_n = [-\frac{1}{2}n^\omega, \frac{1}{2}n^\omega]$ belongs to $\mathbb{D}_k(\omega)$ for every $k\ge 0$. □

The next lemma presents some simple properties of the class $\mathbb{D}_k(\omega)$. Its proof is quite simple and is left for the reader in Exercise 7.

Lemma 1.10 (i) *If f_n belongs to $\mathbb{D}_k(\omega)$, then f_n belongs to $\mathbb{D}_k(\omega')$ for all $\omega' \le \omega$.*

(ii) *If f_n and g_n belong to class $\mathbb{D}_k(\omega)$, and if $H(x,y)$ is a function which is k times continuously differentiable, then $H(f_n, g_n)$ belongs to $\mathbb{D}_k(\omega)$. Consequently if f_n and $g_n \in \mathbb{D}_k(\omega)$, then $f_n + g_n$ and $f_n \times g_n \in \mathbb{D}_k(\omega)$.*

Observe that to prove our general asymptotic expansion, like the one in Theorem 10.6, we must study bounds and Taylor's expansions of the following function

$$h_n(t) = \exp(n(e^{it/\sqrt{n}} - 1 - it/\sqrt{n})) \ . \tag{1.50}$$

One can interpret $h_n(t)$ as the kernel of the Cauchy integral (10.13) (cf. also (10.19)), thus it is not not surprising that this function often appears in our depoissonization theorems. Its properties are discussed in the next lemma.

Lemma 1.11 *The following statements hold:*
(i) *For $t \in [-\pi\sqrt{n}, \pi\sqrt{n}]$ there exists $\mu > 0$ such that $|h_n(t)| \le e^{-\mu t^2}$, where μ is a constant.*
(ii) *For complex t such that $|t| \le Bn^{\frac{1}{6}}$ for some $B > 0$, the sequence of functions $F_n(t) = h_n(t)e^{-t^2/2}$ is bounded and belongs to $\mathbb{D}_k(\frac{1}{6})$ for any integer $k \ge 0$.*

Proof: Part (i) was already proved in Section 10.2.2. Part (ii) is a little more intricate. According to Lemma 10.10, it suffices to prove that the sequence of functions $\log F_n(t) \in \mathbb{D}_k(\frac{1}{6})$, and then refer to the fact that the sequence of exponential of $\log F_n(t)$ still belongs to $\mathbb{D}_k(\frac{1}{6})$. Denoting $e^{ix} - 1 - ix - x^2/2$ by $r(x)$, we observe that the sequence of functions $r(t/\sqrt{n})$ belongs to $\mathbb{D}_k(\frac{1}{2})$ for $t = O(\sqrt{n})$ and any integer $k \ge 1$. Therefore, the ith derivative for $3 \le i \le k$ of $nr(t/\sqrt{n})$ is $O(n^{1-i/2})$, which is $O(n^{-i/6})$ for all $i \ge 3$. In particular, the third derivative is $O(n^{-1/2})$. But we also observe that by successive integrations the first derivative of $nr(t/\sqrt{n})$ is $O(n^{-1/6})$, the second derivative is $O(n^{-1/3})$, and $nr(t/\sqrt{n})$ is $O(1)$, because the first two derivatives of $r(t)$ are zero at $t = 0$ by the construction. Hence, part (ii) follows. ∎

Furthermore, when proving Theorems 10.6, we need an extension of Lemma 8.16 which is presented next.

Lemma 1.12 *For nonnegative H and β, let the sequence of complex functions $F_n(x)$, defined on $x \in [-H\log n, H\log n]$, belong to the class $\mathbb{D}_k(\beta)$ for any $k \ge 1$. Then*

$$\frac{1}{\sqrt{2\pi}} \int_{-H\log n}^{H\log n} F_n(x) e^{-x^2/2} dx = \sum_{i=0}^{k-1} \frac{1}{2^i i!} F_n^{(2i)}(0) + O(n^{-2k\beta}) \ .$$

Proof: By Taylor's expansion $F_n(x) = \sum_{i=0}^{l-1} \frac{x^i}{i!} F^{(i)}(0) + \Delta_n(x)x^l$, where $\Delta_n(x) = O(n^{-l\beta})$ due to $F_n \in \mathbb{D}_l(\beta)$ for any l. Thus

$$\frac{1}{\sqrt{2\pi}} \int_{-H\log n}^{H\log n} F_n(x)e^{-x^2/2}dx = \sum_{i=0}^{l-1} \frac{F_n^{(i)}(0)}{i!} \frac{1}{\sqrt{2\pi}} \int_{-H\log n}^{H\log n} x^i e^{-x^2/2}dx + O(n^{-l\beta})$$

Observe that $F_n^{(i)}(0) = O(n^{-i\beta})$ since $F_n \in \mathbb{D}_l(\beta)$ and $i \le l$. Furthermore, by Lemma 8.16 changing the limits of integration in the above to $\pm\infty$, introduces an error of order $O(e^{-(H\log n)^2/2})$ that decreases faster than any polynomial. Also, by (8.78) of Lemma 8.16 we know that for i even

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^i e^{-x^2/2}dx = \frac{(2m)!}{m!2^m}, \qquad i = 2m,$$

and the integral is zero for i odd. After setting $k = 2l$ and rearranging the above sum, we prove the lemma. ∎

Finally, we are in a position to formulate the so–called **depoissonization tool** theorem. To have our results apply also to the distributional problems discussed in the next section, we consider here a double-index sequence $g_{n,k}$ and a sequence of Poisson transforms

$$\widetilde{G}_k(z) = e^{-z} \sum_{n=0}^{\infty} g_{n,k} \frac{z^n}{n!}.$$

Theorem 1.13 (Jacquet and Szpankowski, 1998) *Let $\widetilde{G}_k(z)$ be the Poisson transform of a sequence $g_{n,k}$ that is assumed to be a sequence of analytical functions for $|z| \le n$. We postulate that the following three conditions simultaneously hold for some $H > 0$, γ, and integer $m \ge 0$:*

(I) *For $z \in \mathcal{S}_\theta$, such that $|z| = n$: $|\widetilde{G}_n(z)| \le F(n)$ for a sequence $F(n)$,*

(L) *The sequence of function $f_n(t) = \widetilde{G}_n(ne^{it/\sqrt{n}})/F(n)$ defined for $t \in [-H\log n, H\log n]$ belongs to the class $\mathbb{D}_m(\gamma)$,*

(O) *For $z \notin \mathcal{S}_\theta$ and $|z| = n$: $|\widetilde{G}_n(z)e^z| \le p(n)e^n F(n)$, where $p(n)$ decays faster than $n^{-\frac{1}{2}-(m+1)\min\{\gamma,\frac{1}{6}\}}$, i.e.,*

$$p(n) = o(n^{-\frac{1}{2}-(m+1)\min\{\gamma,\frac{1}{6}\}}).$$

Then

$$g_{n,n} = \widetilde{G}_n(n) + O(n^{-\min\{\gamma,\frac{1}{6}\}})F(n) \;, \tag{1.51}$$

and more generally for any integer $m \ge 0$

$$g_{n,n} = \sum_{i=0}^{m} \sum_{j=0}^{i+m} b_{ij} n^i \widetilde{G}_n^{(j)}(n) + O(n^{-(m+1)\min\{\gamma,1/6\}})F(n), \tag{1.52}$$

where b_{ij} are defined in (10.40).

Proof. The proof again relies on the Cauchy integral that we split into I_n and E_n as before, where I_n is the integral over t such that ne^{it} is inside the cone $\mathcal{S}_\theta$, and E_n is the integral that has ne^{it} outside the cone. That is:

$$\begin{aligned} I_n &= \omega_n\sqrt{\frac{n}{2\pi}}\int_{-\theta}^{\theta}\widetilde{G}_n(ne^{it})\exp\left(n\left(e^{it}-1-it\right)\right)dt, \\ E_n &= \frac{n!}{2\pi i}\int_{|t|\in[\theta,\pi]}\frac{\widetilde{G}_n(z)e^z}{z^{n+1}}dz \\ &= \omega_n\sqrt{\frac{n}{2\pi}}\int_{|t|\in[\theta,\pi]}\widetilde{G}_n(ne^{it})\exp\left(n\left(e^{it}-1-it\right)\right)dt, \end{aligned}$$

where $\omega_n = n!n^{-n}e^n(2\pi n)^{-1} = 1 + O(n^{-1})$ by the Stirling formula. We estimate the above two integrals. We begin with the latter. Observe that by condition (O) we have $|E_n|/F(n) \le \sqrt{2\pi n}p(n) = o(n^{-k\min\{\gamma,\frac{1}{6}\}})$ for any $k \le m+1$, which is negligible when compared to the error term in (10.51).

The evaluation of I_n is more intricate. First of all, we replace t by $t/\sqrt{n}$ to get

$$\begin{aligned} I_n &= \frac{\omega_n}{\sqrt{2\pi}}\int_{-\sqrt{n}}^{\sqrt{n}}\widetilde{G}_n(ne^{it/\sqrt{n}})\exp\left(n\left(e^{it/\sqrt{n}}-1-it/\sqrt{n}\right)\right)dt \\ &= \frac{\omega_n}{\sqrt{2\pi}}\int_{-\sqrt{n}}^{\sqrt{n}}\widetilde{G}_n(ne^{it/\sqrt{n}})h_n(t)dt, \end{aligned}$$

where $h_n(t)$ is defined in (10.50) and analyzed in Lemma 10.11. We further split the integral I_n into I_n' and I_n'' as follows

$$\begin{aligned} I_n' &= \frac{\omega_n}{\sqrt{2\pi}}\int_{-H\log n}^{H\log n}\widetilde{G}_n(ne^{it/\sqrt{n}})\exp\left(n\left(e^{it/\sqrt{n}}-1-it/\sqrt{n}\right)\right)dt\ , \\ I_n'' &= \frac{\omega_n}{\sqrt{2\pi}}\int_{t\in[-\sqrt{n},-H\log n]}\widetilde{G}_n(ne^{it/\sqrt{n}})\exp\left(n\left(e^{it/\sqrt{n}}-1-it/\sqrt{n}\right)\right)dt \\ &+ \frac{\omega_n}{\sqrt{2\pi}}\int_{t\in[H\log n,\sqrt{n}]}\widetilde{G}_n(ne^{it/\sqrt{n}})\exp\left(n\left(e^{it/\sqrt{n}}-1-it/\sqrt{n}\right)\right)dt\ . \end{aligned}$$

To estimate the second integral I_n'' we use the fact that $|h_n(t)| \le e^{-\mu t^2}$ with $\mu > 0$ for $t \in [-\sqrt{n}, \sqrt{n}]$, as discussed in Lemma 10.11. Thus by Lemma 8.16 and condition (I) we immediately obtain that $I_n''/F(n) = O(e^{-\mu H^2\log^2 n})$, which decays faster than any polynomial.

Now we estimate I_n'. Observe first that since $h_n(t)e^{t^2/2} \in \mathbb{D}_m(\frac{1}{6})$ for any m, and we have $\widetilde{G}_n(ne^{it/\sqrt{n}})/F(n) \in \mathbb{D}_m(\gamma)$, therefore the product

$$R_n(t) = h_n(t)e^{t^2/2}\widetilde{G}_n(ne^{it/\sqrt{n}})/F(n) \in \mathbb{D}_m(\gamma_2),$$

where $\gamma_2 = \min\{\gamma, \frac{1}{6}\}$. In view of this and by Lemma 10.12 we obtain

$$I'_n/F(n) = \frac{\omega_n}{\sqrt{2\pi}} \int_{H \log n}^{H \log n} R_n(t) e^{-t^2/2} dt = \omega_n (F_n(0) + O(n^{-\gamma_2})) ,$$

and consequently $g_n = \widetilde{G}_n(n) + O(n^{-\gamma_2})F(n)$. This proves (10.51).

To prove our general result (10.52), we apply Lemma 10.12 for any k to obtain

$$g_{n,n}/F(n) = \omega_n \sum_{i=0}^{k-1} \frac{1}{2^i i!} R_n^{(2i)}(0) + O(n^{-2k\gamma_2}).$$

Computing explicitly the derivatives of $R_n(t)$ by their actual values (that involve the derivatives of $\widetilde{G}(z)$ at $z = n$), and noting that all odd powers of $n^{-\frac{1}{2}}$ disappear (since we are considering only derivatives of $R_n(t)$ of even order), multiplying by $\omega_n F(n)$, replacing ω_n by the Stirling expansion, we finally obtain

$$g_{n,n} = \sum_{i=0}^{m} \sum_{j=0}^{i+m} b_{ij} n^i \widetilde{G}_n^{(j)}(n) + O(n^{-(m+1)\min\{\gamma, \frac{1}{6}\}})F(n),$$

where we set $m+1 = 2k$. Notice that the terms in the expansion do not involve m since the Stirling expansion and the expansion of Lemma 10.12 do not contain m. ■

Using the depoissonization tool Theorem 10.13, we can now finally prove our remaining theorems by identifying the function $F(n)$ (see condition (I)) and finding the correct value for γ in condition (L) of Theorem 10.13.

We start with the proof of the generalized depoissonization **Theorem 10.6**. First of all, observe that conditions (I) and (O) of Theorem 10.6 imply conditions (I) and (O) of Theorem 10.13 when we set $F(n) = Bn^\beta \Psi(n)$. Thus to complete the proof we must verify condition (L) of Theorem 10.13 and find γ such that

$$f_n(t) = \widetilde{G}_n(ne^{it/\sqrt{n}})n^{-\beta}/\Psi(n) \in \mathbb{D}_m(\gamma)$$

for any integer $m \geq 1$. But by Lemma 10.1(ii), for all integer m and for all z belonging to a smaller cone $\mathcal{S}_{\theta'}$ with $\theta' < \theta$ there exists B_m such that $|\widetilde{G}_n^{(m)}(z)| \leq B_m |z|^{\beta - m} \Psi(|z|)$. After setting $z = ne^{it/\sqrt{n}}$ we see that the sequence of functions $f_n(t)$ belongs to $\mathbb{D}_m(\frac{1}{2})$ for all integer m. Now set $\gamma = \frac{1}{6}$. Then by Theorem 10.13 there is some $m' > m$ such that

$$g_{n,n} = \sum_{i=0}^{m'} \sum_{j=0}^{i+m'} b_{ij} n^i \widetilde{G}_n^{(j)}(n) + O(n^{\beta - m'\gamma})\Psi(n).$$

The last delicate point is to obtain the correct error term in Theorem 10.6. But this can be achieved by setting $m' = \lceil (m+1)/\gamma \rceil$ for a given value m. This will lead an error term equal to $O(n^{\beta-(m+1)})$. Indeed, the additional terms obtained are those with $i > m$ and $j > i+m$. The corresponding coefficients are $b_{ij} n^i \widetilde{G}_n^{(j)}(n)$ with $b_{ij} \neq 0$ for $j \geq 2i$ (we remind the reader that $b_{ij} = 0$ for $j < 2i$, as shown in Section 10.3.1). Then

$$n^{\beta+i-j} \leq n^{\beta+i-2i} = n^{\beta-i} \leq n^{\beta-(m+1)} ,$$

and this proves Theorem 10.6.

Now, we prove the exponential depoissonization **Theorem 10.8**. As before, the proof relies on Theorem 10.13 with $F(n) = \exp(Bn^\beta)$. Conditions (I) and (O) are again easy to verify, so we only need to check condition (L) of Theorem 10.13. That is, we must estimate the growth of $f_n(t) = \widetilde{G}(ne^{it/\sqrt{n}}) \exp(-Bn^\beta)$. But $f_n(t) \in \mathbb{D}(\frac{1}{2} - \beta)$ due to (10.48). To complete the proof, we must establish the error term. Let $\gamma = \min\{\frac{1}{2} - \beta, \frac{1}{6}\}$. An application of Theorem 10.13 leads to the error term $O\left(n^{-m'\gamma} \exp(Bn^\beta)\right)$ for some integer $m' \geq 0$. To establish the right error term $O(n^{-(m+1)(1-2\beta)} \exp(Bn^\beta))$ we follow the same approach as before. We set $m' = \lfloor (m+1)(1-2\beta)/\gamma \rfloor$, which introduces the additional terms $b_{ij} n^i \widetilde{G}^{\langle j \rangle}(n)$ that contribute $O(n^{i-j(1-\beta)} \exp(Bn^\beta))$. But, since $b_{ij} \neq 0$ for $j \geq 2i$,

$$n^{i-j(1-\beta)} \exp(Bn^\beta) \leq n^{i-2i(1-\beta)} \exp(Bn^\beta) \leq n^{-i(1-2\beta)} \exp(Bn^\beta) \leq n^{-m(1-2\beta)} \exp(Bn^\beta) ,$$

and this completes the proof of Theorem 10.8.

1.4 Moments and Limiting Distributions

Depoissonization techniques can also be used to derive limiting distributions. Extension to distributions requires to investigate double-index sequence $g_{n,k}$ (e.g, $g_{n,k} = \Pr\{X_n = k\}$ or $g_{n,k} = \mathbf{E}[e^{tX_n/\sqrt{V_k}}]$ for a sequence V_k). Its Poisson transform is denoted as

$$\widetilde{G}(z,u) = \mathcal{P}(g_{n,k}; z, u) = \sum_{n=0}^{\infty} \frac{z^n}{n!} e^{-z} \sum_{k=0}^{\infty} g_{n,k} u^k.$$

It is often more convenient to investigate the Poisson transform $\widetilde{G}_k(z) = \mathcal{P}(g_{n,k}; z)$ defined as

$$\widetilde{G}_k(z) = \sum_{n=1}^{\infty} g_{n,k} \frac{z^n}{n!} e^{-z},$$

and already discussed in Theorem 10.13. In this section we present the so–called *diagonal depoissonization* result that allows to extract asymptotically $g_{n,n}$ (e.g., $g_{n,n} = \mathbf{E}[e^{tX_n/\mathbf{Var}[X_n]}]$). We start, however, with the relationship between the Poisson and the Bernoulli moments of a random variable.

1.4.1 Moments

Let X_n be a sequence of integer random variable and X_N its corresponding Poisson driven sequence, where N is a Poisson random variable with mean z. Let $\widetilde{G}(z,u) = \mathbf{E}u^{X_N} = \sum_{n=0}^{\infty} \mathbf{E}u^{X_n} \frac{z^n}{n!} e^{-z}$ be its Poisson transform. We introduce also the Poisson mean $\widetilde{X}(z)$ and the Poisson variance $\widetilde{V}(z)$ as

$$\begin{aligned}\widetilde{X}(z) &= \widetilde{G}'_u(z,1) \ , \\ \widetilde{V}(z) &= \widetilde{G}''_u(z,1) + \widetilde{X}(z) - \left(\widetilde{X}(z)\right)^2 ,\end{aligned}$$

where $\widetilde{G}'_u(z,1)$ and $\widetilde{G}''_u(z,1)$ denote respectively the first and the second derivative of $\widetilde{G}(z,u)$ with respect to u at $u=1$.

The next result presents a relationship between the Poisson mean $\widetilde{X}(z)$ and variance $\widetilde{V}(z)$, and the Bernoulli mean $\mathbf{E}[X_n]$ and variance $\mathbf{Var}[X_n]$.

Theorem 1.14 *Let $\widetilde{X}(z)$ and $\widetilde{V}(z)+\widetilde{X}^2(z)$ satisfy condition (O), and $\widetilde{X}(z)$ and $\widetilde{V}(z)$ satisfy condition (I) of Theorem 10.6 with $\beta \le 1$, e.g., $\widetilde{X}(z) = O(z^\beta \Psi(z))$, and $\widetilde{V}(z) = O(z^\beta \Psi(z))$ in a linear cone $\mathcal{S}_\theta$ and appropriate conditions (O) outside the cone, where $\Psi(z)$ is a slowly varying function. Then the following holds:*

$$\begin{aligned}\mathbf{E}[X_n] &= \widetilde{X}(n) - \frac{1}{2} n \widetilde{X}^{\langle 2\rangle}(n) + O(n^{\beta-2}\Psi(n)), && (1.53)\\ \mathbf{Var}[X_n] &= \widetilde{V}(n) - n[\widetilde{X}'(n)]^2 + O(n^{\beta-1}\Psi(n)) && (1.54)\end{aligned}$$

for large n.

Proof. The asymptotic expansion (10.53) follows directly from Theorem 10.6 for $m=1$. To derive (10.54) observe that the Poisson transform of $\mathbf{E}[X_n^2]$ is $\widetilde{V}(z) + (\widetilde{X}(z))^2$, thus again by Theorem 10.6

$$\begin{aligned}\mathbf{E}[X_n^2] &= \widetilde{V}(n) + (\widetilde{X}(n))^2 - \frac{1}{2} n \widetilde{V}^{\langle 2\rangle}(n) - n[\widetilde{X}'(n)]^2 - n\widetilde{X}^{\langle 2\rangle}(n)\widetilde{X}(n) + O(n^{\beta-2}\Psi(n)) \\ &= \widetilde{V}(n) + \left((\widetilde{X}(n))^2 - n\widetilde{X}^{\langle 2\rangle}(n)\widetilde{X}(n) + O(n^{2\beta-2}\Psi(n))\right) - n[\widetilde{X}'(n)]^2 + O(n^{\beta-1}\Psi(n)),\end{aligned}$$

where the last error term is a consequence of $n\widetilde{V}^{\langle 2\rangle}(n) = O(n^{\beta-1}\Psi(n))$. Since $\mathbf{Var}[X_n] = \mathbf{E}[X_n^2] - \mathbf{E}[X_n]^2$, the result follows. ∎

Example 1.10 *I.I.D. Random Variables*

Let $Z_1, \ldots, Z_n$ be a sequence of independently and identically distributed random variables with generating function $P(u) = \mathbf{E}u^{Z_1}$ and mean μ and variance v. The generating function of $X_n = Z_1 + \ldots + Z_n$ is $G_n(u) = (P(u))^n$. Observe that $\mathbf{E}[X_n] = n\mu$ and $\mathbf{Var}[X_n] = nv$. If we consider the Poisson transform of X_n we obtain $\widetilde{G}(z,u) = \exp(z(P(u)-1))$, and $\widetilde{X}(z) = z\mu$, $\widetilde{V}(z) = (\mu^2+v)z$. Thus $\mathbf{Var}[X_n] = \widetilde{V}(n) - n[\widetilde{X}'(n)]^2$, as predicted by Theorem 10.14. □

1.4.2 Limiting Distributions

When dealing with distributions, one must estimate $G_n(u)$ from the Poisson transform $\widetilde{G}(z,u)$. If u belongs to a compact set, then our previous depoissonization results can be directly applied. However, to unify our analysis we present below a general result called the *diagonal depoissonization.*

Theorem 1.15 (Jacquet and Szpankowski, 1998) *Let $\widetilde{G}_k(z)$ be a sequence of Poisson transforms of $g_{n,k}$, which are assumed to be a sequence of entire functions of z. Let the following two conditions hold for some $A>0$, B, $R>0$ and $\alpha>0$, β, and γ:*
(I) *For $z \in S_\theta$ and*

$$|z|>R \quad \Rightarrow \quad |\widetilde{G}_n(z)| \le Bn^\beta |\Psi(n)|, \tag{1.55}$$

where $\Psi(x)$ is a slowly varying function.

(O) *For z outside the linear cone $\mathcal{S}_\theta$*

$$|z|=n \quad \Rightarrow \quad |\widetilde{G}_n(z)e^z| \le n^\gamma \exp(n-An^\alpha)\ , \tag{1.56}$$

Then for large n

$$g_{n,n}=\widetilde{G}_n(n)+O(n^{\beta-1}\Psi(n))\ . \tag{1.57}$$

More generally, for every nonnegative integer m

$$g_{n,n}=\sum_{i=0}^{m}\sum_{j=0}^{i+m} b_{ij}n^i\widetilde{G}_n^{\langle j\rangle}(n)+O(n^{\beta-m-1}\Psi(n)), \tag{1.58}$$

where $\widetilde{G}_n^{(j)}(n)$ denotes the jth derivative of $\widetilde{G}_n(z)$ at $z=n$.

Proof. The proof follows directly from the depoissonization tool Theorem 10.13 and is left for the reader (cf. Exercise 8). ∎

Using this we can prove the following two very useful corollaries.

Corollary 1.16 *Suppose $\widetilde{G}_k(z)=\sum_{n=0}^\infty g_{n,k}\frac{z^n}{n!}e^{-z}$, for k belonging to some set $\mathcal{K}$, are entire functions of z. For some constants A, B, $R>0$, β and $\alpha<1$, let the following two conditions hold uniformly in $k\in\mathcal{K}$:*

(I) *For $z \in \mathcal{S}_\theta$*

$$|z|>R \quad \Rightarrow \quad |\widetilde{G}_k(z)| \le B|z|^\beta\ , \tag{1.59}$$

(O) *For $z \notin \mathcal{S}_\theta$*

$$|z|>R \quad \Rightarrow \quad |\widetilde{G}_k(z)e^z| \le A\exp(\alpha|z|)\ . \tag{1.60}$$

Then uniformly in $k \in \mathcal{K}$

$$g_{n,k}=\widetilde{G}_k(n)+O(n^{\beta-1}) \tag{1.61}$$

and the error estimate does not depend on $\mathcal{K}$.

Proof. The above corollary is a direct consequence of our previous proofs. Nevertheless, we show below how it can be derived from Theorem 10.13. Indeed, let us assume the contrary, i.e., that the thesis of the corollary (10.61) does not hold. In other words, there is a subsequence (n_i, k_i) such that

$$\lim_{i\to\infty}\left|\frac{g_{n_i,k_i}-\widetilde{G}_{k_i}(n_i)}{n_i^{\beta-1}}\right| = \infty\ . \tag{1.62}$$

Observe that n_i cannot be bounded. Indeed, if the subsequence n_i would be bounded, then the uniform boundness of $G_k(z)$ (by our assumption (I)) on any compact set implies that the $g_{n,k}$ are uniformly bounded if n_i are uniformly bounded (it suffice to bound the integrand in $g_{n,k} = \frac{n!}{2i\pi}\oint G_k(z)e^z z^{-n-1}dz$), which contradicts (10.62). So assume now that n_i is unbounded and strictly increasing, and define for a nonnegative integer m

$$\widetilde{H}_m(z) = \begin{cases} 0 & m \neq n_i \\ \widetilde{G}_{k_i}(z) & m = n_i\ . \end{cases}$$

Then $h_{n_i,n_i} = g_{n_i,k_i}$ for all i. Clearly, $\widetilde{H}_m(z)$ satisfies the assumptions of Theorem 10.13 (since it satisfies conditions (I) and (O) of the corollary, which imply conditions (I), (O) and (L) of Theorem 10.13). Thus,

$$h_{n_i,n_i} = g_{n_i,k_i} = \widetilde{G}_{k_i}(n_i) + O(n_i^{\beta-1})$$

which is the desired contradiction. ■

The next corollary is a direct consequence of Corollary 10.16 but since it finds many useful applications, we formulate it separately.

Corollary 1.17 *Let $\widetilde{G}(z,u)$ satisfy the hypothesis of Theorem 10.3, i.e., for some numbers $\theta < \pi/2$, A, B, $\xi > 0$, β, and $\alpha < 1$ (I) and (O) hold for all u in a set $\mathcal{U}$. Then*

$$G_n(u) = \widetilde{G}(n,u) + O(n^{\beta-1}) \tag{1.63}$$

uniformly for $u \in \mathcal{U}$.

Proof. This follows directly from Corollary 10.16. Since the set $\mathcal{K}$ in Corollary 10.16 is arbitrary we can set $\mathcal{K} = \mathcal{U}$, and then $g_{n,u} = \sum_{l\geq 0} g_{n,l}u^l = G_n(u)$ and $\widetilde{G}_u(z) = \widetilde{G}(z,u)$. Thus (10.61) implies (10.63). ■

Example 1.11 *Depth in PATRICIA*

Let us consider the depth in a PATRICIA trie (cf. Section 1.1). The Poisson transform $\widetilde{D}(z,u)$ of the depth satisfies the following functional equation

$$\widetilde{D}(z,u) = u(p\widetilde{D}(zp,u) + q\widetilde{D}(zq,u)) + (1-u)(p\widetilde{D}(zp,u)e^{-qz} + q\widetilde{D}(zq,u)e^{-pz})\ .$$

Due to the factors e^{-qz} and e^{-pz} in front of $\widetilde{D}(z,u)$, the equation is more complicated than the ones we have seen so far. In fact, it does not have a simple explicit solution. Nevertheless, it falls under Theorem 10.5 with $\gamma_1(z) = up + (1-u)pe^{-qz}$, $\gamma_2(z) = uq + (1-u)qe^{-pz}$ and $t(z) = 0$. Furthermore, for $p \neq q$ the asymptotic behavior of $\widetilde{D}(z,u)$ inside a cone $\mathcal{S}_\theta$ $(z \to \infty)$ is determined by the first two terms in the right side of the above equation. Using Mellin transforms Rais *et al.* [346] proved that for large z the Poisson depth is normal (the reader is asked to provide details in Exercise 9), that is, for z large in a cone $\mathcal{S}_\theta$, and $u = e^t$, where t is complex we have:

$$e^{-t\widetilde{X}(z)/\sigma(z)}\widetilde{D}(z, e^{t/\sigma(z)}) = e^{t^2/2}(1 + O(1/\sigma(z))), \tag{1.64}$$

where $\widetilde{X}(z) = O(\log z)$ is the Poisson mean, and $\sigma^2(z) = \widetilde{V}(z) = O(\log z)$ is the Poisson variance.

One must now depoissonize (10.64) in order to establish the central limit law for the Bernoulli model. First of all, an easy application of Theorem 10.14 leads us to $\mathbf{E}[D_n] \sim \widetilde{X}(n)$ and $\mathbf{Var}[D_n] \sim \widetilde{V}(n)$. In fact, from Szpankowski [404] we know that $\mathbf{E}[D_n] = \frac{1}{h}\log n + O(1)$ and $\mathbf{Var}[D_n] = \frac{h^2-h_2}{h^3}\log n + O(1)$, where $h = -p\log p - q\log q$ is the entropy, and $h_2 = p\log^2 p + q\log^2 q$. Thus we only need to depoissonize (10.64) in order to obtain the limiting distribution. We first observe, as in [346], that $\widetilde{D}(z,u) = O(z^\varepsilon)$ for any $\varepsilon > 0$ inside a cone $\mathcal{S}_\theta$ $(\theta < \pi/2)$. This can be proved by simple induction. With the help of Theorem 10.5 we verify conditions (I) and (O) of Corollary 10.17 that finally yields:

Theorem 1.18 (Rais, Jacquet and Szpankowski, 1993) *For complex t*

$$e^{-t\mathbf{E}[D_n]/\sqrt{\mathbf{Var}[D_n]}}\mathbf{E}\left[e^{tD_n/\sqrt{\mathbf{Var}[D_n]}}\right] = e^{t^2/2}(1 + O(1/\sqrt{\log n}))\ ,$$

that is, $(D_n - \mathbf{E}[D_n])/\sqrt{\mathbf{Var}[D_n]}$ converges in distribution and in moments to the standard normal distribution.

Other examples will be discussed in the application section below. □

1.5 Applications

We discuss here three applications. First, we deal with the asymptotic distribution of the height in an incomplete trie associated with the leader election algorithm. Then we tackle the average and the variance of the depth in generalized b-digital search trees for two probabilistic models, namely, for biased memoryless and Markovian sources. The depth of b-DST was already discussed in Section 9.4.1 for the *unbiased* memoryless model. Extension to the biased memoryless model will require a new methodology, that of poissonization and depoissonization. Finally, further extension to the Markovian model will force us to consider systems of functional equations. This last application turns out to be the most sophisticated research problem of this book.

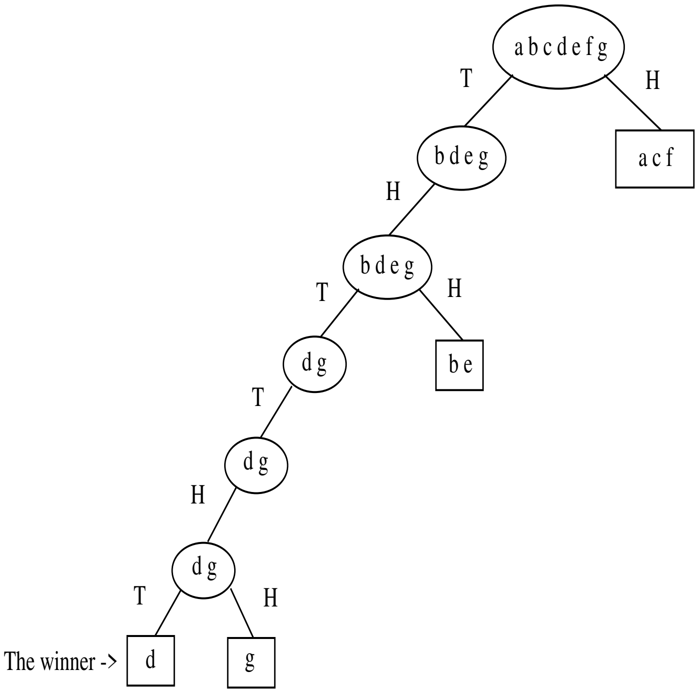

Figure 1.4: An incomplete trie for the elimination process starting with 7 players.

1.5.1 Leader Election Algorithm

This analysis is adopted from Fill *at al.* [122]. The following elimination process has several applications, such as the election of a leader in a computer network. A group of n people play a game to identify a winner by tossing fair coins. All players who throw heads are losers; those who throw tails remain candidates and flip their coins again. The process is repeated until a single winner is identified. If at any stage all remaining candidates throw heads, the tosses are deemed inconclusive and all remaining players participate in the next round of coin tossing.

We investigate the distribution of the height of a random incomplete trie, the discrete structure that underlies the elimination process described above. This binary *incomplete* tree can be described as follows. At the root of the tree we have one node labeled with all participants. After all the participants flip their coins for the first time, losers (if any) are placed in a *leaf* node that is attached to the root as a right child, and all candidate winners are placed in a node that is attached as a left child. Leaf nodes are terminal nodes that are not developed any further. The process repeats recursively on every left child until a single winner is identified. Figure 10.4 illustrates the discrete structure underlying the elimination process.

Our goal is to find the distribution of the height H_n of the associated incomplete trie.

We first use poissonization and then depoissonize the result to obtain exact and the limiting distribution of H_n. For each $n \geq 0$, let

$$G_n(u) := \sum_{k=0}^{\infty} \Pr\{H_n = k\} u^k$$

denote the probability generating function for the fixed-population height H_n (i.e., in the Bernoulli model). The starting point of our analysis is the following recurrence for $G_n(u)$

$$G_n(u) = u2^{-n} \sum_{k=1}^{n} \binom{n}{k} G_k(u) + uG_n(u)2^{-n}, \tag{1.65}$$

which is valid for $n \geq 2$, with the boundary values $G_0(u) = G_1(u) = 1$. This is a standard equation for the depth in digital trees that we have already seen before, except that the associated trie is incomplete so the term $G_{n-k}(u)$ does not appear. The additive term $G_n(u)2^{-n}$ accounts for the fact that when all plays throw heads, then all of them still participate in the game (we must find a leader!).

We now consider the Poisson transform $\widetilde{G}(z,u) = \sum_{n=0}^{\infty} e^{-z} G_n(u) \frac{z^n}{n!}$ of $G_n(u)$, that satisfies the following functional equation

$$\widetilde{G}(z,u) = u(1 + e^{-z/2})\widetilde{G}\left(\frac{z}{2}, u\right) + e^{-z}\left[(1+z)(1-u) - ue^{z/2}\right]. \tag{1.66}$$

To handle this recurrence, we introduce

$$\widetilde{H}(z,u) = \frac{\widetilde{G}(z,u)}{1 - e^{-z}},$$

and obtain

$$\widetilde{H}(z,u) = u\widetilde{H}\left(\frac{z}{2}, u\right) + R(z,u), \tag{1.67}$$

where

$$R(z,u) := \frac{(1+z)(1-u) - ue^{z/2}}{e^z - 1}.$$

The recurrence (10.67) can be solved by direct iteration, as we saw in Section 7.6.1, to obtain

$$\widetilde{H}(z,u) = \sum_{k=0}^{\infty} R\left(\frac{z}{2^k}, u\right) u^k. \tag{1.68}$$

We now start the process of depoissonizing the last equation so that Corolarry 10.16 can be applied. For this we need information about $\widetilde{G}(z,u)$ as $z \to \infty$ in a linear cone and u in a compact set. Certainly, the Mellin transform is the right tool since the sum in (10.68) is a harmonic sum and Theorem 9.1 applies. Simple algebra (with the help of Table 10.1), shows that the Mellin transform $\widetilde{H}^*(s,u)$ of $\widetilde{H}(z,u)$ is

$$\widetilde{H}^*(s,u) = \Gamma(s)\,\zeta(s) + \frac{(1-u)\,\Gamma(s+1)\,\zeta(s+1)}{1-2^s u}$$

for $1 < \Re(s) < -\log|u|$. We further assume that $|u| < \frac{1}{2}$. Since we are concerned here with the distribution function of H_N, where N is a Poisson random variable, rather than its probability mass function, our interest centers on the transform

$$\frac{\widetilde{H}^*(s,u)}{1-u} = \frac{\Gamma(s)\,\zeta(s)}{1-u} + \frac{\Gamma(s+1)\,\zeta(s+1)}{1-2^s u}.$$

This expression can be expanded immediately in a power series in u yielding

$$[u^k]\left(\frac{\widetilde{H}^*(s,u)}{1-u}\right) = \Gamma(s)\,\zeta(s) + 2^{ks}\Gamma(s+1)\,\zeta(s+1).$$

Inverting the transform gives

$$\frac{\widetilde{H}(z,u)}{1-u} = \sum_{k=0}^{\infty} u^k \left[\frac{1}{e^z-1} + \frac{z/2^k}{\exp(z/2^k)-1}\right]$$

for $z > 0$. But

$$\sum_{k=0}^{\infty} \mathsf{Pr}\{H_{N(z)} \le k\} u^k = \frac{\widetilde{G}(z,u)}{1-u} = \frac{(1-e^{-z})\,\widetilde{H}(z,u)}{1-u},$$

where $N(z)$ is a Poisson distributed random variable with mean z. For each $k = 0,1,2,\ldots$ and $z > 0$ we have

$$\mathsf{Pr}\{H_{N(z)} \le k\} = e^{-z} + (1-e^{-z})\,\frac{z/2^k}{\exp(z/2^k)-1}. \tag{1.69}$$

The relation (10.69) can be manipulated to get exact distribution. For any $z > 0$,

$$\begin{aligned}
\mathsf{Pr}\{H_{N(z)} \le k\} &= e^{-z}\left[1 + \frac{z}{2^k}\frac{e^z-1}{\exp(z/2^k)-1}\right] \\
&= e^{-z}\left[1 + \frac{z}{2^k}\sum_{j=0}^{2^k-1}\exp(jz/2^k)\right] \\
&= e^{-z}\left[1 + \frac{1}{2^k}\sum_{j=0}^{2^k-1}\sum_{n=1}^{\infty}\frac{(j/2^k)^{n-1}z^n}{(n-1)!}\right] \\
&= e^{-z}\left[1 + \sum_{n=1}^{\infty}\frac{z^n}{n!}\left(\frac{n}{2^{kn}}\sum_{j=0}^{2^k-1} j^{n-1}\right)\right].
\end{aligned}$$

We give the exact distribution in Theorem 10.19 below.

But we are really after the limiting distribution, if it exists, of the Bernoulli version of the height, that is, H_n. We use the depoissonization Corollary 10.16. Let

$$p_{k,n} := \Pr\{H_n \le k\} \tag{1.70}$$

and then by (10.69) the Poisson transform $\widetilde{P}_k(z)$ of $p_{k,n}$ is

$$\widetilde{P}_k(z) := e^{-z} + (1 - e^{-z}) \frac{z/2^k}{\exp(z/2^k) - 1}. \tag{1.71}$$

Although up until now the right side of (10.71) has arisen only as $\Pr\{H_{N(z)} \le k\}$ for real $z \ge 0$, note that it defines an entire function of the complex variable z (by analytic continuation or the extension of the Mellin transform to complex plane as discussed in Section 9.3).

We must verify conditions (I) and (O) of Corollary 10.16. We may appeal to Theorem 10.5, however, for a change we check the conditions directly. To verify condition (I), we first observe that if $0 \ne v \in \mathcal{S}_\theta$, then $|v| \le \Re v / \cos\theta$ and

$$|e^v - 1| \ge |e^v| - 1 = \exp(\Re v) - 1,$$

so that

$$\left|\frac{v}{e^v - 1}\right| \le (\sec\theta) \frac{\Re v}{\exp(\Re v) - 1} \le \sec\theta.$$

Thus for $0 \ne z \in \mathcal{S}_\theta$ we have

$$\begin{aligned}
|\widetilde{P}_k(z)| &\le |e^{-z}| + |1 - e^{-z}| \left|\frac{z/2^k}{\exp(z/2^k) - 1}\right| \\
&\le |e^{-z}| + (\sec\theta)\, |1 - e^{-z}| \\
&\le \sec\theta + (1 + \sec\theta)\, |e^{-z}| \\
&\le \sec\theta + (1 + \sec\theta) \exp(-\Re z) \\
&\le 1 + 2\sec\theta.
\end{aligned}$$

To verify condition (O), we first observe that if $v \notin \mathcal{S}_\theta$, then $\Re v \le |v| \cos\theta$. Thus for $0 \ne z \notin \mathcal{S}_\theta$, we have from (10.71)

$$\begin{aligned}
|\widetilde{P}_k(z) e^z| &\le \left|1 + \frac{z}{2^k} \times \frac{e^z - 1}{\exp(z/2^k) - 1}\right| \\
&\le 1 + |z| \times 2^{-k} \sum_{r=0}^{2^k - 1} \left|\exp(rz/2^k)\right|
\end{aligned}$$

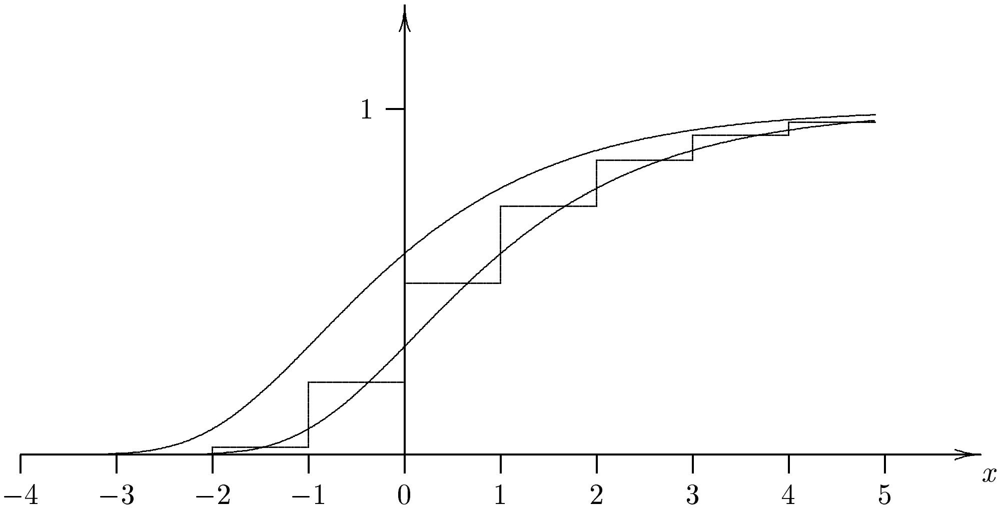

Figure 1.5: The distribution function of $H_{20} - \lfloor \log_2 20 \rfloor$ and the two continuous extremes.

$$\begin{aligned}
&\leq 1 + |z| \times 2^{-k} \sum_{r=0}^{2^k-1} \exp\left(r2^{-k}|z|\cos\theta\right) \\
&\leq 1 + |z| \int_0^1 \exp(x|z|\cos\theta)\,dx \\
&= 1 + (\sec\theta)\left[\exp(|z|\cos\theta) - 1\right] \\
&\leq (\sec\theta)\exp(|z|\cos\theta).
\end{aligned}$$

This verifies both of the conditions in Corollary 10.16 with $\beta = 0$, hence

$$\mathsf{Pr}\{H_n \leq j\} = p_{j,n} = \widetilde{P}_j(n) + O\left(n^{-1}\right) \tag{1.72}$$

holds uniformly for integers $j \geq 0$. But from (10.71)

$$\widetilde{P}_j(n) = e^{-n} + (1 - e^{-n})\,\frac{n/2^j}{\exp(n/2^j) - 1}.$$

Setting $j = \lfloor \log_2 n \rfloor + k$, we find that

$$\widetilde{P}_j(n) = e^{-n} + (1 - e^{-n})\,\frac{2^{\langle \log_2 n\rangle - k}}{\exp\left(2^{\langle \log_2 n\rangle - k}\right) - 1} = \frac{2^{\langle \log_2 n\rangle - k}}{\exp\left(2^{\langle \log_2 n\rangle - k}\right) - 1} + O\left(e^{-n}\right) \tag{1.73}$$

holds, uniformly in all integers k ($\geq -\lfloor \log_2 n \rfloor$). We recall that $\langle \log_2 n \rangle = \log_2 n - \lfloor \log_2 n \rfloor$.

We have thus proved the following result.

Theorem 1.19 (Fill, Mahmoud and Szpankowski, 1996) *Consider an incomplete binary trie with $n \geq 1$, and let B_j denote the jth Bernoulli number.*

(i) *For any integer $k \geq 0$,*

$$\Pr\{H_n \leq k\} = \frac{n}{2^{kn}} \sum_{j=0}^{2^k-1} j^{n-1} = \sum_{j=0}^{n-1} \binom{n}{j} \frac{B_j}{2^{kj}}. \tag{1.74}$$

(ii) *Uniformly over all integers k,*

$$\Pr\{H_n \leq \lfloor \log_2 n \rfloor + k\} = \frac{2^{\langle \log_2 n \rangle - k}}{\exp(2^{\langle \log_2 n \rangle - k}) - 1} + O\left(\frac{1}{n}\right)$$

as $n \to \infty$.

From part (ii) of the above theorem we conclude that the limiting distribution of $H_n - \lfloor \log_2 n \rfloor$ does not exist (due to the term $\langle \log_2 n \rangle$). This is illustrated in Figure 10.5.

1.5.2 Depth in a b-Digital Search Tree for the Biased Memoryless Model

We discuss here the same data structures as in Section 9.4.1, that is, b-digital search trees. However, this time we analyze it under the *biased* memoryless model, that is, for $p \neq q = 1-p$. We concentrate on the depth D_m in a b-DST built over a *fixed* number, say m, of independent binary strings generated in which "0" and "1" occur respectively with probability p and $q = 1 - p$. Let $\bar{B}_m^k$ be the *expected* number of strings at level k of a randomly built b-digital search tree. Since $\Pr\{D_m = k\} = \bar{B}_m^k / m$ (cf. Section 8.7.3), one can alternatively study the average $\bar{B}_m^k$ which is also called *the average profile.* Let $B_m(u) = \sum_{k \geq 0} \bar{B}_m^k u^k$ be the generating function of the average profile. As in Section 9.4.1, we argue that the average profile satisfies the following recurrence for all $m \geq 0$

$$B_{m+b}(u) = b + u \sum_{i=0}^{m} \binom{m}{i} p^i q^{m-i} \left(B_i(u) + B_{m-i}(u)\right), \tag{1.75}$$

with the initial conditions

$$B_i(u) = i \quad \text{for} \quad i = 0, 1, \ldots, b-1 \ .$$

For $b > 1$ this recurrence does not have a closed–form solution. Therefore, we find an approach other than the one used in Section 9.4.1 for the unbiased model. We poissonize the model and study the Poisson transform

$$\widetilde{B}(u, z) = \sum_{i=0}^{\infty} B_i(u) \frac{z^i}{i!} e^{-z} \tag{1.76}$$

of $B_m(u)$. We do so since the recurrence transforms to a slightly more manageable differential–functional equation, namely:

$$\left(1+\frac{\partial}{\partial z}\right)^b \widetilde{B}(u,z) = b + u\left(\widetilde{B}(u,pz)+\widetilde{B}(u,qz)\right),$$

where $(1+\frac{\partial}{\partial z})^b f(z) := \sum_{i=0}^{b}\binom{b}{i}\frac{\partial^i f(z)}{\partial z^i}$. We shall study $\widetilde{B}(u,z)$ for $z\to\infty$ in a cone around the real axis and u in a compact set around $u=1$.

Our goal is to prove the following result.

Theorem 1.20 (Louchard, Szpankowski and Tang, 1999) *Let D_m be the typical depth in a b-digital search tree built over m statistically independent strings generated by a biased memoryless source. Then*

$$\begin{aligned}
\mathbf{E}[D_m] &= \frac{1}{h}\log m + \frac{1}{h}\left(\frac{h_2}{2h}+\gamma-1-H_{b-1}-\Delta(b,p)+\delta_1(m,b)\right)\\
&+ O\left(\frac{\log m}{m}\right)\\
\mathbf{Var}[D_m] &= \frac{h_2-h^2}{h^3}\log m + O(1),
\end{aligned}$$

where $h=-p\log p - q\log q$ is the entropy of the Bernoulli(p) distribution, $h_2 = p\log^2 p + q\log^2 q$, $\gamma = 0.577\ldots$ is the Euler constant, while $H_{b-1}=\sum_{i=1}^{b-1}\frac{1}{i}$, $H_0 = 0$ are harmonic numbers. The constant $\Delta(b,p)$ can be computed as follows

$$\Delta(b,p) = \sum_{n=2b+1}^{\infty}\bar{f}_n\sum_{i=1}^{b}\frac{(i+1)b!}{(b-i)!n(n-1)\cdots(n-i-1)} < \infty,$$

where $\bar{f}_n$ is given recursively by

$$\begin{cases}
f_{m+b} = m + \sum_{i=0}^{m}\binom{m}{i}p^i q^{m-i}(f_i+f_{m-i}), \qquad m>0,\\
f_0 = f_1 = \ldots = f_b = 0,\\
\bar{f}_{m+b} = f_{m+b} - m > 0, \qquad m\ge 1.
\end{cases}\tag{1.77}$$

Finally, $\delta_1(x,b)$ is a fluctuating function with a small amplitude (cf. (10.89)) when $(\log p)/(\log q)$ is rational, and $\lim_{x\to\infty}\delta_1(x,b)=0$ otherwise.

The rest of this subsection is devoted to proving Theorem 10.20. We apply the Mellin transform and depoissonization methods. Observe that $B_m(1)=m$, $\mathbf{E}[D_m]=B_m'(1)/m$, and $B_m''(1)/m = \mathbf{E}[D_m(D_m-1)] = \mathbf{Var}[D_m]-\mathbf{E}[D_m]+\mathbf{E}[D_m]^2$. Thus

$$\mathbf{Var}[D_m] = \frac{B_m''(1)}{m}+\frac{B_m'(1)}{m}-\left(\frac{B_m'(1)}{m}\right)^2.$$

Let $\widetilde{X}(z) := \widetilde{B}_u(1,z)$ and $\widetilde{W}(z) := \widetilde{B}_{uu}(1,z)$ be, respectively, the mean and the second factorial moment of the depth in the Poisson model. Using (10.76), we obtain

$$\left(1+\frac{\partial}{\partial z}\right)^b \widetilde{X}(z) = z + \widetilde{X}(pz) + \widetilde{X}(qz) \ , \tag{1.78}$$

$$\left(1+\frac{\partial}{\partial z}\right)^b \widetilde{W}(z) = 2\left(\widetilde{X}(pz) + \widetilde{X}(qz)\right) + \left(\widetilde{W}(pz) + \widetilde{W}(qz)\right) \ . \tag{1.79}$$

We plan to apply the Mellin transform to find the asymptotics of $\widetilde{X}(z)$ and $\widetilde{W}(z)$ as $z \to \infty$ (first as $x = \Re(z) \to \infty$ and then we analytically continue to the complex plane). Due the appearance of the derivative in (10.78) – (10.79), we rather do not find directly the Mellin transforms $X(s)$ and $Y(s)$ of $\widetilde{X}(z)$ and $\widetilde{W}(z)$, but shall seek $X(s)$ and $Y(s)$ in the following forms

$$X(s) = \mathcal{M}[\widetilde{X}(z); s] = \Gamma(s)\gamma(s) \ , \tag{1.80}$$

$$Y(s) = \mathcal{M}[\widetilde{W}(z); s] = \Gamma(s)\beta(s), \tag{1.81}$$

where $\gamma(s)$ and $\beta(s)$ are to be computed. They exist in the proper strips, as proved below.

Lemma 1.21 . *The following is true:* (i) *$X(s)$ exists for $\Re(s) \in (-b-1, -1)$, and $Y(s)$ is defined for $\Re(s) \in (-2b-1, -1)$.*

(ii) *Furthermore, $\gamma(-1-i) = 0$ for $i = 1, \ldots, b-1$, $\gamma(-1-b) = (-1)^{b+1}$, and $\beta(-1-i) = 0$ for $i = 1, \ldots, b$, and $\gamma(s)$ has simple poles at $s = -1, 0, 1, \ldots$.*

Proof: By recurrence (10.75), we have $B_i(u) = i$ for $i = 0, 1, \ldots, b$, and thus $B_i(u) = b + (i-b)u$ for $i = 1+b, \ldots, 2b$. Taking derivatives, we obtain $\frac{\partial B_i(u)}{\partial u} = 0$ for $i = 0, 1, \ldots, b$ and $\frac{\partial B_i(u)}{\partial u} = i - b$ for $i = b, 1+b, \ldots, 2b$. Furthermore, the second derivative becomes $\frac{\partial^2 B_i(u)}{\partial u^2} = 0$ for $i = 0, 1, \ldots, 2b$. Hence, for $z \to 0$

$$\begin{aligned} \widetilde{X}(z) &= \left(z^{b+1}/(b+1)! + 2z^{b+2}/(b+2)! + 3z^{b+3}/(b+3)! + O(z^{b+4})\right) e^{-z} \ , \\ &= z^{b+1}/(b+1)! + O(z^{b+2}) \quad as \quad z \to 0 \\ \widetilde{W}(z) &= O(z^{2b+1}) \quad as \quad z \to 0 \ . \end{aligned}$$

On the other hand, for $z \to \infty$ we conclude from (10.78) and (10.79) that $\widetilde{X}(z) = O(z \log z)$ and $\widetilde{W}(z) = O(z \log^2 z)$. Thus the first part of the lemma is proven. Part (ii) follows directly from the lemma below and (10.86). ∎

Lemma 1.22 *Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of real numbers having the Poisson transform $\widetilde{F}(z) = \sum_{n=0}^{\infty} f_n \frac{z^n}{n!} e^{-z}$, which is an entire function. Furthermore, let its Mellin transform $F(s)$ have the following factorization*

$$F(s) = \mathcal{M}[\widetilde{F}(z); s] = \Gamma(s)\gamma(s).$$

Assume that $F(s)$ exists for $\Re(s) \in (-2,-1)$, and that $\gamma(s)$ is analytic for $\Re(s) \in (-\infty,-1)$. Then

$$\gamma(-n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_k, \quad \text{for} \quad n \geq 2. \tag{1.82}$$

Proof: Let sequence $\{g_n\}_{n=0}^{\infty}$ be such that $\widetilde{F}(z) = \sum_{n=0}^{\infty} g_n \frac{z^n}{n!}$, that is,

$$g_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f_k \ , \qquad n \geq 0 \ .$$

Define for some fixed $M \geq 2$, the function $\widetilde{F}_M(z) = \sum_{n=0}^{M-1} g_n \frac{z^n}{n!}$. Due to our assumptions, we can continue $F(s)$ analytically to the whole complex plane except $s = -2, -3, \ldots$. In particular, for $\Re(s) \in (-M, -M+1)$ we have

$$F(s) = \mathcal{M}[\widetilde{F}(z) - \widetilde{F}_M(z); s].$$

(The above is true since a polynomial in z, such as $\widetilde{F}_M(z)$, can only shift the fundamental strip of the Mellin transform but cannot change its value, as we saw in Section 9.2.) As $s \to -M$, due to the factorization $F(s) = \Gamma(s)\gamma(s)$, we have

$$F(s) = \frac{1}{s+M} \frac{(-1)^M}{M!} \gamma(-M) + O(1) \ ;$$

thus by the inverse Mellin transform, we have

$$\widetilde{F}(z) - \widetilde{F}_M(z) = \frac{(-1)^M}{M!} \gamma(-M) z^M + O(z^{M+1}) \quad \text{as} \quad z \to 0 \ . \tag{1.83}$$

But

$$\widetilde{F}(z) - \widetilde{F}_M(z) = \sum_{i=M}^{\infty} g_n \frac{z^n}{n!} = g_M \frac{z^M}{M!} + O(z^{M+1}) \ . \tag{1.84}$$

Comparing (10.83) and (10.84), shows that $\gamma(-M) = (-1)^M g_M = \sum_{k=0}^{M} \binom{M}{k} (-1)^k f_k$ for $M \geq 2$. ■

Now we are in a position to compute the Mellin transforms of $\widetilde{X}(z)$ and $\widetilde{W}(z)$. From (10.78)–(10.81) we obtain

$$\begin{aligned}
\sum_{i=0}^{b} \binom{b}{i} (-1)^i \gamma(s-i) &= (p^{-s} + q^{-s})\gamma(s) \ , \\
\sum_{i=0}^{b} \binom{b}{i} (-1)^i \beta(s-i) &= 2(p^{-s} + q^{-s})\gamma(s) + (p^{-s} + q^{-s})\beta(s) \ ,
\end{aligned}$$

and, by Lemma 10.21, $\gamma(s)$ exists at least for $\Re(s) \in (-b-1,-1)$, while $\beta(s)$ is well defined in the strip $\Re(s) \in (-2b-1,-1)$. To simplify the above, we define, for any function $g(s)$,

$$\widehat{g}(s) = \sum_{i=1}^{b} \binom{b}{i} (-1)^{i+1} g(s-i), \tag{1.85}$$

provided $g(s-1), \ldots, g(s-b)$ exist. Then

$$\begin{aligned} \gamma(s) &= \frac{1}{1-p^{-s}-q^{-s}} \sum_{i=1}^{b} \binom{b}{i} (-1)^{i+1} \gamma(s-i) = \frac{1}{1-p^{-s}-q^{-s}} \widehat{\gamma}(s) && (1.86)\\ \beta(s) &= \frac{1}{1-p^{-s}-q^{-s}} \widehat{\beta}(s) + \frac{2(p^{-s}+q^{-s})}{(1-p^{-s}-q^{-s})^2} \widehat{\gamma}(s) \ . && (1.87) \end{aligned}$$

Let s_k, $k = 0, \pm 1, \pm 2, \ldots$, be roots of $1 - p^{-s} - q^{-s} = 0$. Observe that $s_0 = -1$. We need to know precisely the locations of the roots s_k. Fortunately, we already studied this problem in Lemma 8.22. At $s = s_k$ we have

$$\frac{1}{1-p^{-s}-q^{-s}} = -\frac{1}{h(s_k)} \frac{1}{s-s_k} + \frac{h_2(s_k)}{2h^2(s_k)} + O(s-s_k), \tag{1.88}$$

where

$$\begin{aligned} h(t) &= -p^{-t} \log p - q^{-t} \log q, \\ h_2(t) &= p^{-t} \log^2 p + q^{-t} \log^2 q. \end{aligned}$$

Expanding $\Gamma(s)\widehat{\gamma}(s)$ around $s = s_k$, we find that

$$\Gamma(s)\widehat{\gamma}(s) = \Gamma(s_k)\widehat{\gamma}(s_k) + \left(\Gamma(s_k)\widehat{\gamma}'(s_k) + \Gamma'(s_k)\widehat{\gamma}(s_k)\right)(s-s_k) + O((s-s_k)^2) \ .$$

Therefore, since $X(s) = \Gamma(s)\gamma(s) = \frac{1}{1-p^{-s}-q^{-s}} \Gamma(s)\widehat{\gamma}(s)$, we obtain

$$\begin{aligned} X(s) &= -\frac{1}{s-s_k} \frac{\Gamma(s_k)}{h(s_k)} \widehat{\gamma}(s_k) + \frac{h_2(s_k)}{2h^2(s_k)} \Gamma(s_k)\widehat{\gamma}(s_k) \\ &- \frac{1}{h(s_k)} \left(\Gamma(s_k)\widehat{\gamma}'(s_k) + \Gamma'(s_k)\widehat{\gamma}(s_k)\right) + O(s-s_k) \ . \end{aligned}$$

In a similar manner, from (10.87) we have

$$\begin{aligned} Y(s) &= \frac{2\Gamma(s_k)\widehat{\gamma}(s_k)}{h^2(s_k)} \frac{1}{(s-s_k)^2} + \left(\frac{2\Gamma'(s_k)\widehat{\gamma}(s_k)}{h^2(s_k)} - \frac{\Gamma(s_k)}{h(s_k)} \widehat{\beta}(s_k) \right. \\ &- \left. 2\Gamma(s_k) \frac{h_2(s_k) - h^2(s_k)}{h^3(s_k)} \widehat{\gamma}(s_k) - \frac{2\Gamma(s_k)\widehat{\gamma}'(s_k)}{h^2(s_k)} \right) \frac{1}{s-s_k} + O(1). \end{aligned}$$

On the other hand, from (10.88) and (10.86) at $s = s_0 = -1$, we find that

$$\begin{aligned}\gamma(s) &= -\frac{1}{h}\frac{1}{s+1} + \frac{h_2}{2h^2} - \frac{\widehat{\gamma}'(-1)}{h} + O(s+1), \\ \beta(s) &= \frac{2}{h^2}\frac{1}{(s+1)^2} + \left(-2\frac{h_2-h^2}{h^3} + 2\widehat{\gamma}'(-1)\frac{1}{h^2}\right)\frac{1}{s+1} + O(1)\ .\end{aligned}$$

In the above, we used the fact that $\widehat{\gamma}(-1) = 1$ and $\widehat{\beta}(-1) = 0$, which follows directly from Lemma 10.21. But $\Gamma(s) = -\frac{1}{s+1} + (\gamma - 1) + O(s+1)$; hence the Laurent expansion of $X(s)$ at $s = -1$ is

$$X(s) = \Gamma(s)\gamma(s) = \frac{1}{h}\frac{1}{(s+1)^2} - \left(\frac{h_2}{2h^2} - \frac{1}{h}\widehat{\gamma}'(-1) + \frac{\gamma-1}{h}\right)\frac{1}{s+1} + O(1)\ .$$

We now use the reverse mapping Theorem 9.2 to find asymptotically $\widetilde{X}(z)$ and $\widetilde{W}(z)$ for $z \to \infty$ in a cone $\mathcal{S}_\theta$ (we again use either analytic continuation or the extension to the complex plane of the Mellin transform as discussed in Section 9.3). However, to estimate the error term we note that $\gamma(s)$ has additional simple poles at $s = 0, 1, \ldots$. The pole at $s = 0$ is a double pole of $X(s) = \Gamma(s)\gamma(s)$, and thus its contribution to $\widetilde{X}(z)$ is $O(\log z)$. Putting everything together, we finally arrive at

$$\begin{aligned}\widetilde{X}(z) &= \frac{1}{h}z\log z + \left(\frac{h_2}{2h^2} - \frac{1}{h}\widehat{\gamma}'(-1) + \frac{\gamma-1}{h}\right)z \\ &+ \sum_{k\neq 0}\frac{\Gamma(s_k)\widehat{\gamma}(s_k)}{h(s_k)}z^{-s_k} + O(\log z).\end{aligned} \tag{1.89}$$

Similarly, at $s = -1$,

$$Y(s) = -\frac{2}{h^2}\frac{1}{(s+1)^3} + \frac{2}{h}\left(\frac{h_2-h^2}{h^2} - \frac{1}{h}\widehat{\gamma}'(-1) + \frac{\gamma-1}{h}\right)\frac{1}{(s+1)^2} + O\left(\frac{1}{s+1}\right).$$

In addition, there is a double pole at $s = 0$, and hence by the reverse mapping Theorem 9.2 and Lemma 8.22 we obtain

$$\begin{aligned}\widetilde{W}(z) &= \frac{1}{h^2}z\log^2 z + \frac{2}{h}\left(\frac{h_2-h^2}{h^2} - \frac{1}{h}\widehat{\gamma}'(-1) + \frac{\gamma-1}{h}\right)z\log z \\ &+2\sum_{k\neq 0}\frac{\Gamma(s_k)\widehat{\gamma}(s_k)}{h^2(s_k)}z^{-s_k}\log z + O(z)\end{aligned}$$

for $z \to \infty$, where $O(z)$ error term comes from the term $O((s+1)^{-1})$. This formula will allow us to infer the asymptotics of the variance of D_m.

The final step is depoissonization. We shall use Theorem 10.3. To verify conditions (I) and (O) we need a slight extension of Theorem 10.5. Condition (I) follows from (10.89), thus we need only verify (O). We consider only $\widetilde{X}(z)$. Let $X(z) = \widetilde{X}(z)e^z$. Then the functional equation (10.78) transforms into

$$X^{(b)}(z) = X(zp)e^{zq} + X(zq)e^{zp} + ze^z,$$

where $X^{(b)}(z)$ denotes the bth derivative of $X(z)$. Observe that the above equation can be integrated to yield

$$X(z) = \underbrace{\int_0^z \int_0^{w_1} \cdots \int_0^{w_{b-1}}}_{b \text{ times}} \left(X(w_1 p)e^{w_1 q} + X(w_1 q)e^{w_1 p} + w_1 e^{w_1}\right) dw_b \cdots dw_2 dw_1, \qquad (1.90)$$

where the integration is along lines in the complex plane.

We now prove that $|X(z)| \le e^{\alpha|z|}$ for $z \notin \mathcal{S}_\theta$ for $\alpha < 1$. We use induction over *increasing domains* as discussed in Theorem 10.5. That is, for all positive integers $m \ge 1$ and constants $\xi, \delta > 0$, let

$$\mathcal{D}_m = \{z = \rho e^{i\vartheta} : \rho \in [\xi\delta, \xi\nu^{-m}], \quad 0 \le \vartheta < 2\pi\},$$

where $\max\{p,q\} \le \nu < 1$ and $\delta \le \min\{p,q\}$. As we have already seen, such a representation enjoys the property that if $z \in \mathcal{D}_{m+1} - \mathcal{D}_m$ then $zp, zq \in \mathcal{D}_m$ provided that $|z| \ge \xi$, which is assumed to hold. To carry out the induction, we first define $\bar{\mathcal{D}}_m = \mathcal{D}_m \cap \bar{\mathcal{S}}_\theta$, where $\bar{\mathcal{S}}_\theta$ denotes points in the complex plane outside $\mathcal{S}_\theta$. Since $X(z)$ is bounded for $z \in \bar{\mathcal{D}}_1$, the initial step of induction holds. Let us now assume that for some $m > 1$ and for $z \in \bar{\mathcal{D}}_m$ we have $|X(z)| \le e^{\alpha|z|}$ with $\alpha < 1$. We intend to prove that $|X(z)| \le e^{\alpha|z|}$ for $z \in \bar{\mathcal{D}}_{m+1}$. Indeed, let $z \in \bar{\mathcal{D}}_{m+1}$. If also $z \in \bar{\mathcal{D}}_m$, then the proof is complete. Let us assume that $z \in \bar{\mathcal{D}}_{m+1} - \bar{\mathcal{D}}_m$. Since then $zp, zq \in \bar{\mathcal{D}}_m$, we can use our induction hypothesis together with the integral equation (10.90) to obtain the following estimate for $|z| > \xi$, where ξ is sufficiently large:

$$|X(z)| \le |z|^{b+1}\left(e^{|z|(p\alpha + q\cos\theta)} + e^{|z|(q\alpha + p\cos\theta)} + e^{|z|\cos\theta}\right).$$

Let us now define $1 > \alpha > \cos\theta$ such that the following three inequalities simultaneously hold:

$$\begin{aligned} |z|^b e^{|z|(p\alpha + q\cos\theta)} &\le \frac{1}{3}e^{\alpha|z|}, \\ |z|^b e^{|z|(q\alpha + p\cos\theta)} &\le \frac{1}{3}e^{\alpha|z|}, \\ |z|^{b+1} e^{|z|\cos\theta} &\le \frac{1}{3}e^{\alpha|z|}. \end{aligned}$$

Then for $z \in \bar{\mathcal{D}}_{m+1}$ we have $|X(z)| \le e^{\alpha|z|}$, as needed to verify condition (O) of the depoissonization lemma.

In view of the above, we can apply the depoissonization Theorem 10.3 to $\widetilde{X}(z)$ and $\widetilde{W}(z)$ and obtain

$$\begin{aligned}
\mathbf{E}[D_m] &= \frac{\widetilde{X}(m)}{m} + O\left(\frac{\log m}{m}\right) \\
&= \frac{1}{h}\log m + \frac{h_2}{2h^2} - \frac{1}{h}\widehat{\gamma}'(-1) + \frac{\gamma - 1}{h} + \sum_{k\neq 0} \frac{\Gamma(s_k)\widehat{\gamma}(s_k)}{h(s_k)} m^{-1-s_k} + O\left(\frac{\log m}{m}\right),
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{E}[D_m(D_m-1)] &= \frac{\widetilde{W}(m)}{m} + O(1) = \frac{1}{h^2}\log^2 m + 2\frac{1}{h}\left(\frac{h_2 - h^2}{h^2} - \frac{1}{h}\widehat{\gamma}'(-1) + \frac{\gamma-1}{h}\right)\log m \\
&+ 2\sum_{k\neq 0} \frac{\Gamma(s_k)\widehat{\gamma}(s_k)}{h^2(s_k)} m^{-1-s_k}\log m + O(1) \ .
\end{aligned}$$

After computing $\mathbf{E}[D_m]^2$ we arrive at

$$\begin{aligned}
\mathbf{Var}[D_m] &= ED_m(D_m-1) + ED_m - (ED_m)^2 \\
&= \frac{h_2 - h^2}{h^3}\log m + 2\sum_{k\neq 0}\frac{\Gamma(s_k)\widehat{\gamma}(s_k)}{h(s_k)}\left(\frac{1}{h(s_k)} - \frac{1}{h}\right) m^{-1-s_k}\log m + O(1) \ .
\end{aligned}$$

If $\Re(s_k) = -1$ for all k, then by Lemma 8.22 one can prove that $h(s_k) = h$ (cf. Exercise 19). If $\Re(s_k) > -1$, then $m^{-1-s_k}\log m = o(1)$. Therefore, $\mathbf{Var}[D_m] = \frac{h_2 - h^2}{h^3}\log m + O(1)$. From Lemma 8.22 we know that $\Re(s_k) = -1$ whenever $(\log p)/(\log q)$ is rational, and otherwise $\Re(s_k) > -1$.

To complete the proof of Theorem 10.20 we evaluate the constant $\widehat{\gamma}'(-1)$. In passing we point out that the second-order term of $\mathbf{E}[D_m]$ plays an important role in some applications (e.g., the computation of the average code redundancy for the standard Lempel-Ziv scheme [296], and for k-resilient Lempel-Ziv codes, introduced recently by Storer and Reif [395]). Therefore, knowing its value, or providing a numerical algorithm to compute it, is of prime interest. But the second-order term depends on $\widehat{\gamma}'(-1)$, which can be also expressed as

$$\widehat{\gamma}'(-1) = \sum_{i=1}^{b}\binom{b}{i}(-1)^{i+1}\gamma'(-1-i),$$

where $\gamma(s)\Gamma(s) = \mathcal{M}[\widetilde{X}(z); s]$. Define the sequence $\{f_m\}_{m=0}^{\infty}$ from $\tilde{X}(z) = \sum_{m=0}^{\infty} f_m \frac{z^m}{m!}e^{-z}$. Since clearly $f_m = \mathbf{E}[D_m]$, it satisfies recurrence (10.77) of Theorem 10.20.

Assume first that $b > 1$. To compute $\widehat{\gamma}'(-1)$ we must find $\gamma(s)$ in terms of computable quantities, such as f_n. We proceed as follows:

$$\gamma(s) = \frac{1}{\Gamma(s)}\mathcal{M}[\sum_{n=b+1}^{\infty} f_n\frac{z^n}{n!}e^{-z}; s] = \sum_{n=b+1}^{\infty}\frac{f_n}{n!}\frac{\Gamma(s+n)}{\Gamma(s)}$$

$$= \sum_{n=b+1}^{\infty} \frac{f_n}{n!} s(s+1)\cdots(s+n-1) \ .$$

We assume that $\Re(s) \in (-b-1,-1)$ to assure the existence of the Mellin transform and the convergence of the above series. Then one easily derives

$$\gamma'(s) = \sum_{n=b+1}^{\infty} \frac{f_n}{n!} s(s+1)\cdots(s+n-1) \sum_{i=0}^{n-1} \frac{1}{s+i}, \qquad s \notin \{-2,-3,\ldots,-b\} \ .$$

After some algebra, we arrive at the following:

$$\begin{aligned}
\gamma'(-k) &= (-1)^k \sum_{n=b+1}^{\infty} \frac{f_n}{n!} k!(n-k-1)!, \quad \text{for} \quad k=2,\ldots,b, \\
\gamma'(-b-1) &= (-1)^b H_{b+1} + (-1)^{b+1} \sum_{n=b+2}^{\infty} \frac{f_n}{n!}(b+1)!(n-b-2)! \ .
\end{aligned}$$

To estimate $\widehat{\gamma}'(-1)$ we find, after some tedious algebra, that

$$\widehat{\gamma}'(-1) = \sum_{i=1}^{b} \binom{b}{i} (-1)^{i+1} \gamma'(-i-1) - \frac{1}{b} - \frac{b}{b+1} + A + \Delta(b,p),$$

where

$$\begin{aligned}
\Delta(b,p) &= \sum_{n=b+2}^{\infty} \bar{f}_n \sum_{i=1}^{b} \frac{(i+1)b!}{(b-i)!n(n-1)\cdots(n-i-1)}, \\
A &= \sum_{n=b+2}^{\infty} (n-b) \sum_{i=1}^{b} \frac{(i+1)b!}{(b-i)!n(n-1)\cdots(n-i-1)}.
\end{aligned} \tag{1.91}$$

We recall that $\bar{f}_{m+b} = f_{m+b} - m$ and f_m satisfies (10.77). The above series converge since the summands are $O(\log n/n^2)$. Finally, observe that $\bar{f}_{m+b} = 0$ for $m = 1,2,\ldots,b$ and $\bar{f}_i > 0$ for $i > 2b$; hence

$$\begin{aligned}
\Delta(b,p) &= \sum_{n=b+2}^{\infty} \bar{f}_n \sum_{i=1}^{b} \frac{(i+1)b!}{(b-i)!n(n-1)...(n-i-1)} \\
&= \sum_{n=2b+1}^{\infty} \bar{f}_n \sum_{i=1}^{b} \frac{(i+1)b!}{(b-i)!n(n-1)\cdots(n-i-1)} \ .
\end{aligned}$$

It can be proved that $A = H_b + b(1+b)^{-1}$ (the reader is asked to verify this in Exercise 20), hence $\widehat{\gamma}'(-1) = H_{b-1} + \Delta(b,p)$ as presented in Theorem 10.20.

The case $b=1$ was already discussed in Section 8.7.3, where we found

$$\Delta(1,p)=\theta=-\sum_{k=1}^{\infty}\frac{p^{k+1}\log p+q^{k+1}\log q}{1-p^{k+1}-q^{k+1}}\ .$$

This finally completes the proof of Theorem 10.20 for all b.

1.5.3 Depth in a Digital Search Tree with a Markovian Source

We again analyze the depth in a digital search tree with $b=1$. However, this time the strings are emitted by a Markovian source. More precisely, we assume that m sequences are generated independently by a Markov source over a finite alphabet $\mathcal{A}=\{1,2,\ldots,V\}$ with a positive transition matrix $\mathsf{P}=\{p_{ij}\}_{i,j=1}^{V}$ (i.e., $p_{ij}>0$ for all $i,j\in\mathcal{A}$) and a stationary vector $\boldsymbol{\pi}=(\pi_1,\ldots,\pi_V)$.

We build a digital search tree from m strings. Let I_i be the depth of insertion for the ith string, that is, the path length from the root to the node containing the ith string. As before, the typical depth D_m is defined as

$$\Pr\{D_m=k\}=\frac{1}{m}\sum_{i=1}^{m}\Pr\{I_i=k\},$$

and the *average* profile (in short: profile) $\bar{B}_m^k$ represents the *average* number of nodes at level k of the tree. Observe that $\bar{B}_0^k=0$ for all $k\geq 0$, and for $m\geq 1$

$$\Pr\{D_m=k\}=\frac{\bar{B}_m^k}{m}.$$

This and the definition of the typical depth immediately imply that

$$\Pr\{I_{m+1}=k\}=\bar{B}_{m+1}^k-\bar{B}_m^k, \tag{1.92}$$

with $\Pr\{I_0=0\}=1$ and $\Pr\{I_0=k\}=0$ for all $k\geq 1$. Throughout, we shall work with generating functions of the above quantities, that is,

$$\begin{aligned}
D_m(u) &= \mathbf{E}[u^{D_m}]=\sum_{k\geq 0}\Pr\{D_m=k\}u^k, && D_0(u)=1,\\
I_m(i) &= \mathbf{E}[u^{I_m}]=\sum_{k\geq 0}\Pr\{I_m=k\}u^k, && I_0(u)=1,\\
B_m(u) &= \sum_{k\geq 0}\bar{B}_m^k u^k && B_0(u)=0
\end{aligned}$$

for a complex u such that $|u| < 1$, and the Poisson transforms

$$\begin{aligned}\widetilde{D}(z,u) &= \sum_{m\geq 0} D_m(u)\frac{z^m}{m!}e^{-z},\\ \widetilde{B}(z,u) &= \sum_{m\geq 0} B_m(u)\frac{z^m}{m!}e^{-z},\\ \widetilde{I}(z,u) &= \sum_{m\geq 0} I_m(u)\frac{z^m}{m!}e^{-z}.\end{aligned}$$

Observe that from (10.92), we have

$$\frac{\partial \widetilde{I}(z,u)}{\partial z} + \widetilde{I}(z,u) = \frac{\partial \widetilde{B}(z,u)}{\partial z}.$$

Since $D_m(u) = B_m(u)/m$, we can recover all results on the depth of insertion I_m and on the typical depth, from the average profile $\bar{B}_m^k$. Therefore, hereafter we concentrate on the analysis of the average profile.

To start the analysis, we derive a system of recurrence equations for the generating function of the average profile. Let $B_m^i(u)$ for $i \in \mathcal{A}$ be the ordinary generating function of the average profile when all sequences start with symbol i. Let also $\mathbf{p} = (p_1, \ldots, p_V)$ be the initial probability vector of the underlying Markov chain, that is, $\Pr\{X_0 = i\} = p_i$. (For the stationary Markov chain we have $\mathbf{p} = \boldsymbol{\pi}$.) The probability that the first subtree of the root contains j_1 strings, the second subtree has j_2 strings, and so on until the Vth subtree stores j_V strings is distributed as

$$\binom{m}{j_1, \ldots, j_V} p_1^{j_1} \cdots p_V^{j_V}.$$

But the ith subtree is a digital search tree itself of size j_i containing only those strings that start with symbol i. Hence, its average profile generating function must be $B_{j_i}^i(u)$. This leads to the following recurrence equation, assuming $B_0(u) = 0$

$$B_{m+1}(u) = 1 + u \sum_{|\mathbf{j}|=m} \binom{m}{\mathbf{j}} p_1^{j_1} \cdots p_V^{j_V} \left(B_{j_1}^1(u) + \cdots + B_{j_V}^V(u)\right), \tag{1.93}$$

where $\mathbf{j} = (j_1, \ldots, j_V)$, $|\mathbf{j}| = j_1 + \cdots + j_V$ and for simplicity we write $\binom{m}{\mathbf{j}} = \binom{m}{j_1,\ldots,j_V}$. Clearly, we can set up similar recurrences for the subtrees. That is,

$$B_{m+1}^i(u) = 1 + u \sum_{|\mathbf{j}|=m} \binom{m}{\mathbf{j}} p_{i1}^{j_1} \ldots p_{iV}^{j_V} \left(B_{j_1}^1(u) + \cdots + B_{j_V}^V(u)\right), \qquad \text{for all} \quad i \in \mathcal{A}, \tag{1.94}$$

where $B_0^i(u) = 0$ for $i \in \mathcal{A}$.

Before we present the main result, let us establish some notation. For a complex s we define

$$\mathsf{Q}(s) = \mathsf{I} - \mathsf{P}(s), \quad \text{where} \quad \mathsf{P}(s) = \{p_{ij}^{-s}\}_{i,j=1}^{V},$$

where I is the identity matrix. In addition, we use the standard notation for the entropy of a Markov source, that is,

$$h = -\sum_{i=1}^{V} \pi_i \sum_{j=1}^{V} p_{ij} \ln p_{ij},$$

and for a probability vector $\mathbf{p} = (p_1, \ldots, p_V)$ we define

$$h_{\mathbf{p}} = -\sum_{i=1}^{V} p_i \ln p_i.$$

We also write $\mathbf{p}(s) = [\pi_1^{-s}, \pi_2^{-s}, \cdots, \pi_V^{-s}]$, which becomes $\boldsymbol{\pi}$ when $s = -1$.

Finally, for the matrix $\mathsf{P}(s)$ we define the principal left eigenvector $\boldsymbol{\pi}(s)$, and the principal right eigenvector $\boldsymbol{\psi}(s)$ associated with the largest eigenvalue $\lambda(s)$ as

$$\begin{aligned} \boldsymbol{\pi}(s)\mathsf{P}(s) &= \lambda(s)\boldsymbol{\pi}(s), \\ \mathsf{P}(s)\boldsymbol{\psi}(s) &= \lambda(s)\boldsymbol{\psi}(s), \end{aligned}$$

where $\boldsymbol{\pi}(s)\boldsymbol{\psi}(s) = 1$. Observe that $\boldsymbol{\pi}(-1) = \boldsymbol{\pi} = (\pi_1, \ldots, \pi_V)$, $\boldsymbol{\psi}(-1) = \boldsymbol{\psi} = (1, \ldots, 1)$, and $\lambda(-1) = 1$. For a function, vector or matrix $A(s)$ we write $\dot{A}(s) = \frac{d}{ds}A(s)$ and $\ddot{A}(s) = \frac{d^2}{ds^2}A(s)$. In Exercise 22 the reader is asked to check that

$$\dot{\lambda}(-1) = \boldsymbol{\pi}\dot{\mathsf{P}}(-1)\boldsymbol{\psi} = h, \tag{1.95}$$

$$\ddot{\lambda}(-1) = \boldsymbol{\pi}\ddot{\mathsf{P}}(-1)\boldsymbol{\psi} + 2\dot{\boldsymbol{\pi}}(-1)\dot{\mathsf{P}}(-1)\boldsymbol{\psi} - 2\dot{\lambda}(-1)\dot{\boldsymbol{\pi}}(-1)\boldsymbol{\psi}. \tag{1.96}$$

We now formulate the main result that we derive throughout the rest of this section.

Theorem 1.23 (Jacquet, Szpankowski and Tang, 2000) *Let a digital search tree be built from m strings generated independently by a Markov stationary source with positive transition probabilities $\mathsf{P} = \{p_{ij}\}_{i,j=1}^{V}$.*

(i) [Typical Depth] *For large m the following holds*

$$\begin{aligned} \mathbf{E}[D_m] &= \frac{1}{\dot{\lambda}(-1)}\left(\ln m + \gamma - 1 + \dot{\lambda}(-1) + \frac{\ddot{\lambda}(-1)}{2\dot{\lambda}^2(-1)} - \vartheta - \boldsymbol{\pi}\dot{\boldsymbol{\psi}}(-1) + \delta_1(\ln m)\right) \\ &+ O\left(\frac{\ln m}{m}\right), \end{aligned} \tag{1.97}$$

$$\mathbf{Var}[D_m] = \frac{\ddot{\lambda}(-1) - \dot{\lambda}^2(-1)}{\dot{\lambda}^3(-1)} \ln m + O(1). \tag{1.98}$$

Here $\vartheta = \boldsymbol{\pi}\dot{\mathbf{x}}(-2)$ *and*

$$\begin{aligned}\dot{\mathbf{x}}(-2) &:= \sum_{i=1}^{\infty}\left(\mathsf{Q}^{-1}(-2)\cdots\mathsf{Q}^{-1}(-i)(\mathsf{Q}^{-1}(s))'|_{s=-i-1}\mathsf{Q}^{-1}(-i-2)\cdots\right)\mathsf{K}, &(1.99)\\ \mathsf{K} &= \left(\prod_{i=0}^{\infty}Q^{-1}(-2-i)\right)^{-1}\boldsymbol{\psi}, &(1.100)\end{aligned}$$

where $\boldsymbol{\psi} = (1,\ldots,1)$. *The function* $\delta_1(x)$ *is a fluctuating function with a small amplitude when*

$$\frac{\ln p_{ij} + \ln p_{1i} - \ln p_{1j}}{\ln p_{11}} \in \mathbb{Q}, \qquad i,j = 1,2,\ldots,V, \tag{1.101}$$

where $\mathbb{Q}$ *is the set of rational numbers. If (10.101) does not hold, then* $\lim_{x\to\infty}\delta_1(x) = 0$.

(ii) [Depth of Insertion] *The depth of insertion* I_m *behaves asymptotically as* D_m, *that is,*

$$\begin{aligned}\mathbf{E}[I_m] &= \frac{1}{\dot{\lambda}(-1)}\left(\ln m + \gamma + \dot{\lambda}(-1) + \frac{\ddot{\lambda}(-1)}{2\dot{\lambda}^2(-1)} - \vartheta - \boldsymbol{\pi}\dot{\boldsymbol{\psi}}(-1) + \delta_2(\ln m)\right)\\ &+ O\left(\frac{\ln m}{m}\right)\\ \mathbf{Var}[I_m] &= \mathbf{Var}[D_m] + O(1),\end{aligned}$$

where $\delta_2(x)$ *is a fluctuating function with the same property as* $\delta_1(x)$.

The rest of this section is devoted to proving Theorem 10.23. The generating function $B_m(u)$ of the average profile satisfies (10.93) with the initial vector $\mathbf{p} = \boldsymbol{\pi}$. Observe that the conditional generating functions $B_m^i(u)$ fulfill the system of recurrence equations (10.94). We shall first deal with (10.94). There is no easy way to solve these recurrences. Therefore, we transform them to the Poisson model. Let

$$\widetilde{B}^i(z,u) = \sum_{n=1}^{\infty} B_n^i(u)\frac{z^n}{n!}e^{-z}, \quad i \in \mathcal{A}$$

be the Poisson transform of $B_m^i(u)$. In addition, we shall write $\widetilde{B}_z^i(z,u) := \frac{\partial}{\partial z}\widetilde{B}^i(z,u)$ for the derivative of $\widetilde{B}^i(z,u)$ with respect to z. After some simple algebra, we have the following poissonized differential–functional equations of recurrences (10.93) and (10.94)

$$\widetilde{B}_z(z,u) + \widetilde{B}(z,u) = u[\widetilde{B}^1(u,\pi_1 z) + \cdots + \widetilde{B}^V(u,\pi_V z)] + 1,$$

and

$$\widetilde{B}_z^i(z,u) + \widetilde{B}^i(z,u) = u[\widetilde{B}^1(u,p_{i1}z) + \cdots + \widetilde{B}^V(u,p_{iV}z)] + 1 \qquad \text{for all} \quad i \in \mathcal{A}.$$

We now concentrate on the evaluation of the first two moments of the depth. Thus we need the first two derivatives of $\widetilde{B}(z,u)$ with respect to u at $u=1$. Noting that $\widetilde{B}^i(z,1)=z$, we have

$$
\begin{aligned}
\widetilde{B}_{zu}(z,1)+\widetilde{B}_u(z,1) &= z+[\widetilde{B}_u^1(1,\pi_1 z)+\cdots+\widetilde{B}_u^V(1,\pi_V z)],\\
\widetilde{B}_{zu}^i(z,1)+\widetilde{B}_u^i(z,1) &= z+[\widetilde{B}_u^1(1,p_{11} z)+\cdots+\widetilde{B}_u^V(1,p_{1V} z)], \qquad i\in\mathcal{A},
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{B}_{zuu}(z,1)+\widetilde{B}_{uu}(z,1) &= 2[\widetilde{B}_u^1(1,\pi_1 z)+\cdots+\widetilde{B}_u^V(1,\pi_V z)]\\
&+ [\widetilde{B}_{uu}^1(1,\pi_1 z)+\cdots+\widetilde{B}_{uu}^V(1,\pi_V z)],\\
\widetilde{B}_{zuu}^i(z,1)+\widetilde{B}_{uu}^i(z,1) &= 2[\widetilde{B}_u^1(1,p_{11} z)+\cdots+\widetilde{B}_u^V(1,p_{1V} z)] \qquad i\in\mathcal{A}\\
&+ [\widetilde{B}_{uu}^1(1,p_{11} z)+\cdots+\widetilde{B}_{uu}^V(1,p_{1V} z)]
\end{aligned}
$$

Our goal is to solve asymptotically (as $z\to\infty$ in a cone around $\Re(z)>0$) the above two sets of functional equations using Mellin transforms. A direct solution through the Mellin transform does not work well, as we already saw in the previous section. Therefore, we factorize the Mellin transforms as follows

$$
\begin{aligned}
B_i^*(s) &:= \mathcal{M}[B_u^i(z,1);s]=\Gamma(s)x_i(s), \qquad i\in\mathcal{A}\\
B^*(s) &:= \mathcal{M}[B_u(z,1);s]=\Gamma(s)x(s),\\
C_i^*(s) &:= \mathcal{M}[B_{uu}^i(z,1);s]=\Gamma(s)v_i(s), \qquad i\in\mathcal{A}\\
C^*(s) &:= \mathcal{M}[B_{uu}(z,1);s]=\Gamma(s)v(s),
\end{aligned}
$$

where $x_i(s)$, $x(s)$, $v_i(s)$ and $v(s)$ are unknown. The lemma below establishes the existence of the above Mellin transforms. The reader is asked to prove it in Exercise 23

Lemma 1.24 *The Mellin transforms $B_i^*(s)$, $B^*(s)$ and $C_i^*(s)$, $C^*(s)$ exist for $\Re(s)\in(-2,-1)$. In addition,*

$$x_i(-2)=1, \qquad v_i(-2)=0, \qquad i\in\mathcal{A},$$

and $x(-2)=1$, $v(-2)=0$.

Using properties of Mellin transforms (cf. (M3) of Chapter 9) we obtain

$$
\begin{aligned}
-(s-1)B^*(s-1)+B^*(s) &= B_1^*(s)\pi_1^{-s}+\cdots+B_V^*(s)\pi_V^{-s}, && (1.102)\\
-(s-1)B_i^*(s-1)+B_i^*(s) &= B_1^*(s)p_{11}^{-s}+\cdots+B_V^*(s)p_{1V}^{-s}, \qquad i\in\mathcal{A}, && (1.103)
\end{aligned}
$$

and

$$
\begin{aligned}
-(s-1)C^*(s-1)+C^*(s) &= 2[B_1^*(s)\pi_1^{-s}+\cdots+B_V^*(s)\pi_V^{-s}] && (1.104)\\
&+ [C_1^*(s)\pi_1^{-s}+\cdots+C_V^*(s)\pi_V^{-s}], && (1.105)\\
-(s-1)C_i^*(s-1)+C_i^*(s) &= 2[B_1^*(s)p_{11}^{-s}+\cdots+B_V^*(s)p_{1V}^{-s}] && (1.106)\\
&+ [C_1^*(s)p_{11}^{-s}+\cdots+C_V^*(s)p_{1V}^{-s}], \qquad i\in\mathcal{A}. && (1.107)
\end{aligned}
$$

Now set

$$\begin{aligned}
\mathbf{x}(s) &= (x_1(s), \ldots, x_V(s))^T, \\
\mathbf{v}(s) &= (v_1(s), \ldots, v_V(s))^T, \\
\mathbf{b}(s) &= (B_1^*(s), \ldots, B_V^*(s))^T, \\
\mathbf{c}(s) &= (C_1^*(s), \ldots, C_V^*(s))^T,
\end{aligned}$$

where T denotes transpose. Using $\Gamma(s) = (s-1)\Gamma(s-1)$, we can rewrite the above equations as

$$\begin{aligned}
\mathbf{x}(s) - \mathbf{x}(s-1) &= \mathsf{P}(s)\mathbf{x}(s), \\
\mathbf{v}(s) - \mathbf{v}(s-1) &= 2\mathsf{P}(s)\mathbf{x}(s) + \mathsf{P}(s)\mathbf{v}(s).
\end{aligned}$$

Thus

$$\mathbf{x}(s) = \mathsf{Q}^{-1}(s)\mathbf{x}(s-1) = \left(\prod_{i=0}^{\infty} \mathsf{Q}^{-1}(s-i)\right)\mathsf{K}, \tag{1.108}$$

$$\mathbf{v}(s) = 2\mathsf{Q}^{-1}(s)\mathsf{P}(s)\mathbf{x}(s) + \mathsf{Q}^{-1}(s)\mathbf{v}(s-1). \tag{1.109}$$

We recall that $\mathsf{Q} = \mathsf{I} - \mathsf{P}$, I is the identity matrix, and K is defined in (10.100). The formula for K follows from Lemma 10.24 (i.e., $\mathbf{x}(-2) = (1, \ldots, 1)^T$) and (10.108).

Thus far we have obtained the Mellin transforms of the conditional generating functions $\widetilde{B}_i(z,1)$. In order to obtain the composite Mellin transform $B^*(s)$ and $C^*(s)$ of $\widetilde{B}_u(z,1)$ and $\widetilde{B}_{uu}(z,1)$, we use to (10.102) and (10.104) and, after some algebra, we finally obtain

$$B^*(s) = \mathbf{p}(s)\mathbf{b}(s) + \Gamma(s)x(s-1), \tag{1.110}$$

$$C^*(s) = 2\mathbf{p}(s)\mathbf{b}(s) + \mathbf{p}(s)\mathbf{c}(s) + \Gamma(s)v(s-1), \tag{1.111}$$

where $\mathbf{p}(s) = (\pi_1^{-s}, \ldots, \pi_V^{-s})$. We shall see that the dominant asymptotics of $B^*(s)$ and $C^*(s)$ follow from the asymptotics of $\mathbf{b}(s)$ and $\mathbf{c}(s)$, and these depend on the singularities of $\mathsf{Q}(s)$, which we discuss next.

We now prove the following lemma that characterizes the location of singularities of $\mathsf{Q}(s)$. It extends the Jacquet-Schachinger Lemma 8.22 to Markovian sources.

Lemma 1.25 (Tang 1996) *Let $\mathsf{Q}(s) = \mathsf{I} - \mathsf{P}(s)$ and $\mathsf{P}(s) = \{p_{ij}^{-s}\}_{i,j\in\mathcal{A}}$. Let s_l denote the singularities of $\mathsf{Q}(s)$, where $l \in \mathbb{Z}$ is an integer. Then:*

(i) *Matrix $\mathsf{Q}(s)$ is nonsingular for $\Re(s) < -1$, and $s_0 = -1$ is a simple pole.*

(ii) *Matrix $\mathsf{Q}(s)$ has simple poles on the line $\Re(s) = -1$ if and only if*

$$\frac{\ln p_{ij} + \ln p_{1i} - \ln p_{1j}}{\ln p_{11}} \in \mathbb{Q}, \qquad i,j \in \mathcal{A}, \tag{1.112}$$

where $\mathbb{Q}$ is the set of rational numbers. These poles have the following representation

$$s_l = -1 + l\theta i,$$

where

$$\theta = \frac{n_1}{n_2}\left|\frac{2\pi}{\ln p_{11}}\right|,$$

for integers n_1, n_2 such that $\left\{\left|\frac{n_1}{n_2 \ln p_{11}}(\ln p_{ij} + \ln p_{1i} - \ln p_{1j})\right|\right\}_{ij=1}^{V}$ is a set of relative primes.

(iii) *Finally,*

$$\mathsf{Q}(-1 + l\theta i) = \mathsf{E}^{-l}\mathsf{Q}(-1)\mathsf{E}^{l},$$

where $\mathsf{E} = \mathrm{diag}(1, e^{\theta_{12} i}, \ldots, e^{\theta_{1V} i})$ is the diagonal matrix with $\theta_{ik} = -\theta \ln p_{ik}$.

Proof. Observe that for $\Re(s) < -1$,

$$|1 - p_{ii}^{-s}| \geq 1 - |p_{ii}^{-s}| > 1 - p_{ii} = \sum_{j\neq i} p_{ij} \geq \sum_{j \neq i} |p_{ij}^{-s}|, \tag{1.113}$$

hence $\mathsf{Q}(s)$ is a strictly diagonally dominant matrix, and therefore nonsingular (cf. [198, 323]). We proceed with the proof of part (ii) of the lemma. For $b \neq 0$ such that $\mathsf{Q}(-1+bi)$ is singular, let $\mathbf{x} = [x_1, x_2, ..., x_V]^T \neq 0$ be a solution of $\mathsf{Q}(-1 + bi)\mathbf{x} = 0$, where

$$\mathsf{Q}(-1+bi) = \begin{bmatrix} 1 - p_{11}e^{\xi_{11} i} & -p_{12}e^{\xi_{12} i} & \ldots & -p_{1V}e^{\xi_{1V} i} \\ -p_{21}e^{\xi_{21} i} & 1 - p_{22}e^{\xi_{22} i} & \ldots & -p_{2V}e^{\xi_{2V} i} \\ \vdots & \vdots & \ddots & \vdots \\ -p_{j1}e^{\xi_{j1} i} & -p_{j2}e^{\xi_{j2} i} & \ldots & -p_{jV}e^{\xi_{jV} i} \\ \vdots & \vdots & \ddots & \vdots \\ -p_{V1}e^{\xi_{V1} i} & -p_{V2}e^{\xi_{V2} i} & \ldots & 1 - p_{VV}e^{\xi_{VV} i} \end{bmatrix}$$

with $\xi_{ik} = -b \ln p_{ik}$. Without loss of generality, suppose $|x_1| = \max\{|x_1|, |x_2|, ..., |x_V|\} \neq 0$ (since $\mathsf{Q}(-1 + bi)$ is singular). Then

$$(1 - p_{11}e^{\xi_{11} i})x_1 - p_{12}e^{\xi_{12} i}x_2 - \cdots - p_{1V}e^{\xi_{1V} i}x_V = 0,$$

implies that

$$1 - p_{11}e^{\xi_{11} i} = p_{12}e^{\xi_{12} i}x_2/x_1 + \cdots + p_{1V}e^{\xi_{1V} i}x_V/x_1.$$

But as in (10.113)

$$|1 - p_{11}e^{\xi_{11} i}| \geq 1 - p_{11},$$

and

$$|p_{12}e^{\xi_{12}i}x_2/x_1 + ... + p_{1V}e^{\xi_{1V}i}x_V/x_1| \le p_{12} + \cdots + p_{1V} = 1 - p_{11}.$$

Thus

$$1 - p_{11}e^{\xi_{11}i} = 1 - p_{11},$$
$$p_{12}e^{\xi_{12}i}x_2/x_1 + \cdots + p_{1V}e^{\xi_{1V}i}x_V/x_1 = p_{12} + \cdots + p_{1V}.$$

This implies that

$$e^{\xi_{11}i} = e^{\xi_{1i}i}x_i/x_1 = 1,$$

and $|x_i| = |x_j|$ for any $i, j = 1, 2, \ldots, V$, so that $e^{\xi_{jj}i} = 1$ for all j. Define now ξ_i such that $x_i/x_1 = e^{-\xi_{1i}i} = e^{\xi_i i}$. Then

$$-p_{j1}e^{\xi_{j1}i} - p_{j2}e^{\xi_{j2}i}e^{\xi_2 i} - \cdots - p_{j(j-1)}e^{\xi_{j(j-1)}i}e^{\xi_{j-1}i} + (1 - p_{jj})e^{\xi_j i} - \cdots - p_{jV}e^{\xi_{jV}i}e^{\xi_V i} = 0$$

for any $1 \le j \le V$. Note that since

$$-p_{j1} - p_{j2} - \cdots - p_{j(j-1)} + 1 - p_{jj} - \cdots - p_{jV} = 0$$

we must have $e^{\xi_{ji}i}e^{\xi_i i}e^{-\xi_j i} = 1$, and thus

$$e^{\xi_{ji}i} = e^{(\xi_j - \xi_i)i}.$$

Hence, $-b(\ln p_{ji} + \ln p_{1j} - \ln p_{1i}) = 2\pi n_{ji}$ for some integer n_{ji}, and as a consequence $(\ln p_{ij} + \ln p_{1i} - \ln p_{1j})/\ln p_{11}$ is rational for any $i, j = 1, 2, \ldots, V$.

To prove the second part of (ii), suppose b is such that $|\frac{b}{2\pi}(\ln p_{ji} + \ln p_{1j} - \ln p_{1i})|$ are integers for any $i, j = 1, 2, \ldots, V$. Then

$$\mathsf{Q}(-1+bi) = \begin{bmatrix} 1 - p_{11} & -p_{12}e^{(\xi_1-\xi_2)i} & \cdots & -p_{1V}e^{(\xi_1-\xi_V)\mathbf{i}} \\ -p_{21}e^{(\xi_2-\xi_1)i} & 1 - p_{22} & \cdots & -p_{2V}e^{(\xi_2-\xi_V)i} \\ \vdots & \vdots & \ddots & \vdots \\ -p_{i1}e^{(\xi_i-\xi_1)i} & -p_{i2}e^{(\xi_i-\xi_2)i} \cdots & -p_{iV}e^{(\xi_i-\xi_V)i} & \\ \vdots & \vdots & \ddots & \vdots \\ -p_{V1}e^{(\xi_V-\xi_1)i} & -p_{V2}e^{(\xi_V-\xi_2)i} & \cdots & 1 - p_{VV} \end{bmatrix}$$
$$= [\mathrm{diag}(1, e^{-\xi_2}, e^{-\xi_3}, \ldots, e^{-\xi_V})]^{-1}\mathsf{Q}(-1)\mathrm{diag}(1, e^{-\xi_2}, e^{-\xi_3}, \ldots, e^{-\xi_V}).$$

Since $\mathsf{Q}(-1)$ is singular, so is $\mathsf{Q}(-1+bi)$. Hence $s = -1 + bi$ is a pole of $\mathsf{Q}(s)$ if and only if $|\frac{b}{2\pi}(\ln p_{ji} + \ln p_{1j} - \ln p_{1i})|$ are integers for any $i, j = 1, 2, \ldots, V$. Since $\{|\frac{\theta}{2\pi}(\ln p_{ij} + \ln p_{1i} -$

$\ln p_{1j})|\}_{ij=1}^V$ is a set of relative primes, we have $b = l\theta$ for some integer l. Thus part (ii) is proved. Part (iii) follows from our previous analysis. ■

Observe that for the memoryless case we have $p_{ji} = \pi_i$, and condition (10.112) becomes $\frac{\ln \pi_i}{\ln \pi_j} \in \mathbb{Q}$ for all i, j. This agrees with Lemma 8.22.

Now, we are prepared to prove (10.97)–(10.98) of Theorem 10.23. We first consider the mean. The starting point is (10.108), which we rewrite as

$$\mathbf{x}(s) = \mathsf{Q}^{-1}\mathbf{x}(s-1) = \sum_{k=0}^{\infty} \mathsf{P}^k(s)\mathbf{x}(s-1).$$

We denote by $\lambda(s), \mu_2(s), \ldots, \mu_V(s)$ the eigenvalues of $\mathsf{P}(s)$. By the Perron-Frobenius Theorem 4.5 we know that $|\lambda(s)| > |\mu_1(s)| \geq \cdots \geq |\mu_V(s)|$ (recall that we assumed that $p_{ij} > 0$ for all $i, j \in \mathcal{A}$). The corresponding left eigenvectors are $\boldsymbol{\pi}(s), \boldsymbol{\pi}_2(s), \ldots, \boldsymbol{\pi}_V(s)$ while the right eigenvectors are $\boldsymbol{\psi}(s), \boldsymbol{\psi}_2(s), \ldots, \boldsymbol{\psi}_V(s)$. We write here $\langle \mathbf{x}, \mathbf{y} \rangle$ for the scalar product of the vectors $\mathbf{x}$ and $\mathbf{y}$.

Using the spectral representation of $\mathsf{P}(s)$ (cf. Table 4.1) we obtain

$$\mathsf{P}^k(s)\mathbf{x}(s-1) = \lambda^k(s)\langle \boldsymbol{\pi}(s), \mathbf{x}(s-1)\rangle \boldsymbol{\psi}(s) + \sum_{i=2}^{V} \mu_i^k(s) \langle \boldsymbol{\pi}_i(s), \mathbf{x}(s-1)\rangle \boldsymbol{\psi}_i(s).$$

Thus $\mathbf{b}(s) = \Gamma(s)\mathbf{x}(s)$ becomes

$$\mathbf{b}(s) = \frac{\Gamma(s)\langle \boldsymbol{\pi}(s), \mathbf{x}(s-1)\rangle \boldsymbol{\psi}(s)}{1 - \lambda(s)} + \sum_{i=2}^{V} \frac{\Gamma(s)\langle \boldsymbol{\pi}_i(s), \mathbf{x}(s-1)\rangle \boldsymbol{\psi}_i(s)}{1 - \mu_i(s)}. \tag{1.114}$$

To obtain the leading order asymptotics of $B^*(s) = \mathbf{p}(s)\mathbf{b}(s) + \Gamma(s)x(s-1)$ (cf. (10.110)), we need Laurent's expansion of the above around the roots of $\lambda(s) = -1$. Observe that the second term of (10.114) contributes $o(m)$ since $\lambda(s)$ is the largest eigenvalue, hence we ignore this term. To simplify the presentation, we only deal here with the root $s_0 = -1$. The other singularities are handled in the same way as in the previous section. We use our previous expansions for $\mathbf{x}(s-1)$ and $\Gamma(s)$ together with

$$\begin{aligned} \frac{1}{1-\lambda(s)} &= \frac{-1}{\dot{\lambda}(-1)} \frac{1}{s+1} + \frac{\ddot{\lambda}(-1)}{2\dot{\lambda}^2(-1)} + O(s+1), \\ \boldsymbol{\psi}(s) &= \boldsymbol{\psi} + \dot{\boldsymbol{\psi}}(-1)(s+1) + O((s+1)^2). \end{aligned}$$

This finally leads to

$$\begin{aligned} B^*(s) &= \frac{-1}{\dot{\lambda}(-1)} \frac{1}{(s+1)^2} \\ &+ \frac{1}{s+1} \left(\frac{\langle \boldsymbol{\pi}, \dot{\mathbf{x}}(-2)\rangle}{\dot{\lambda}(-1)} - \frac{\gamma - 1}{\dot{\lambda}(-1)} + \frac{\langle \mathbf{p}(-1), \dot{\boldsymbol{\psi}}(-1)\rangle}{\dot{\lambda}(-1)} + \frac{\ddot{\lambda}(-1)}{2\dot{\lambda}^2(-1)} - 1 \right) + O(1). \end{aligned}$$

After finding the inverse Mellin transform and depoissonizing, we obtain (10.97) for the mean $\mathbf{E}[D_m]$.

Finally, we turn our attention to the second factorial moment and the variance. We need to study $\mathbf{c}(s) = \Gamma(s)\mathbf{v}(s)$, where $\mathbf{v}(s) = 2\mathsf{Q}^{-1}(s)\mathsf{P}(s)\mathbf{x}(s) + \mathsf{Q}^{-1}(s)\mathbf{v}(s-1)$. As before, we obtain

$$\mathbf{c}(s) = \frac{2\Gamma(s)\langle\boldsymbol{\pi}(s), \mathbf{x}(s-1)\rangle\langle\boldsymbol{\pi}(s), \mathsf{P}(s)\boldsymbol{\psi}(s)\rangle\boldsymbol{\psi}(s)}{(1-\lambda(s))^2} + O\left((1-\lambda(s))^{-1}\right).$$

A calculation similar to that for $\mathbf{b}(s)$ leads to

$$\begin{aligned}\mathbf{c}(s) &= \frac{-2}{\dot{\lambda}^2(-1)}\frac{1}{(s+1)^3} \\ &+ \frac{1}{(s+1)^2}\left(\frac{\ddot{\lambda}(-1)}{2\dot{\lambda}^3(-1)} + 2\frac{\gamma - 1 - \langle\boldsymbol{\pi}, \dot{\mathbf{x}}(-2)\rangle - \langle\mathbf{p}(-1), \dot{\boldsymbol{\psi}}(-1)\rangle - \dot{\lambda}(-1)}{\dot{\lambda}^2(-1)}\right) \\ &+ O\left(\frac{1}{s+1}\right).\end{aligned}$$

This is sufficient to prove (10.98), after some tedious algebra that was done using MAPLE. Details of the proof can be found in Jacquet *et al.* [218].

1.6 Extensions and Exercises

1.1 Prove properties (P1)-(P3) of the Poisson process.

1.2 Let S_n be the size of a trie built over n independent strings generated by a memoryless source. Define $S_n(u) = \mathbf{E}[u^{S_n}]$ and let $\widetilde{S}(z,u)$ be the Poisson transform of $S_n(u)$. Prove that

$$\widetilde{S}(z,u) = u\widetilde{S}(zp,u)\widetilde{S}(zq,u) + (1-u)(1+z)e^{-z}.$$

1.3 Let $g_n = n(n-1)\cdots(n-k+1)$. First show that $\widetilde{G}(z) = z^k$. Next prove directly the general depoissonization result (as in Theorem 10.6) in this case, and show that the coefficients b_{ij} defined in (10.40) are equal to

$$b_{ij} = \frac{1}{j!}s_2(j,i),$$

where $s_2(n,k)$ are related to Stirling's number of first kind, and defined via

$$\sum_{n,k} s_2(n,k)t^n u^k/n! = e^{-tu}(1+t)^u$$

(cf. Comtet [69]).

1.4 Consider
$$g_{n,k} = 1 + \frac{n!}{(k+\log k)^n}.$$
Find its Poisson transform and decide whether the analytic depoissonization will work in this case. Explain your answer. If you cannot justify it, read carefully Theorem 10.13 and see if you can apply it to this case.

1.5 ⚠ (Jacquet and Szpankowski 1989) Consider the functional equation discussed in Example 5, that is,
$$\widetilde{C}(z) = \left(1 + (1+z/2)e^{-z/2}\right)\widetilde{C}(z/2).$$
Establish asymptotics (10.36) of $\widetilde{C}(z)$ as $z \to \infty$ in a cone $\mathcal{S}_\theta$ around the real axis. In particular, compute the constant D that appears in the asymptotic expansion of $\widetilde{C}(z)$.

1.6 Consider the Laplace transform $g^\star(s)$ of a function $g(x)$. Extend the definition of the Laplace transform to the complex plane (assuming $g(x)$ can be analytically continued to $g(z)$). The reader should use arguments similar to those we adopted Section 9.3, where we extended Mellin transforms to the complex plane. In particular, prove that if $g(z)$ is an analytic continuation of $g(x)$ in a cone $\mathcal{S}_\theta$, where $\theta < \theta_0$, then the inverse Laplace transform $g^\star(s)$ of the function $g(z)$ exists in a *bigger* cone $\mathcal{S}_{\theta+\pi/2}$ for all $\theta < \theta_0$, *provided* that $g(z) = O(z^\beta)$ in the cone $\mathcal{S}_{\theta_0}$.

1.7 Prove Lemma 10.10.

1.8 Provide details of the proof of the diagonal depoissonization Theorem 10.15.

1.9 ⚠ (Rais, Jacquet and Szpankowski 1993) Prove that the depth in a PATRICIA trie in the Poisson model is normally distributed, that is, establish (10.64).

1.10 ⚠ (Flajolet 1999) Use singularity analysis to derive the asymptotic expansion for Shannon entropy h_n of the binomial distribution, that is,
$$h_n = -\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \log\left(\binom{n}{k} p^k q^{n-k}\right).$$

1.11 Let
$$h_n(t) = \exp(n(e^{it/\sqrt{n}} - 1 - it/\sqrt{n}))$$

(cf. Lemma 10.11). Prove that for all nonnegative integers k there exist ν_k such that for all $t \in [-\log n, \log n]$,

$$h_n(t) = e^{-t^2/2}\left(1 + \sum_{i=3}^{2k}\sum_{j=1}^{k} \xi_{ij} t^i n^{j-i/2} + O(\nu_k \log^{3(k+1)} n^{-(k+1)/2})\right),$$

where the coefficients ξ_{ij} are defined as

$$\sum_{ij\geq 0} \xi_{ij} x^i y^j = \exp(y(e^{ix} - 1 - ix + \frac{1}{2}x^2)).$$

1.12 ⚠ (Jacquet and Szpankowski 1998) The purpose of this exercise is to extend the general depoissonization Theorem 10.6 to the so–called polynomial cones $\mathcal{C}(D,\delta)$ defined as follows:

$$\mathcal{C}(D,\delta) = \{z = x + iy : \ |y| \leq Dx^\delta \ , \quad 0 < \delta \leq 1 \ , \quad D > 0\} \ .$$

Observe that when $\delta = 1$ the polynomial cone becomes a linear cone. Prove the following extension of Theorem 10.6.

Theorem 1.26 *Consider a polynomial cone $\mathcal{C}(D,\delta)$ with $1/2 < \delta \leq 1$. Let the following two conditions hold for some numbers A, B, $R > 0$ and $\alpha > 0$, β, and γ:*
(I) *For $z \in \mathcal{C}(D,\delta)$*

$$|z| > R \quad \Rightarrow \quad |\widetilde{G}(z)| \leq B|z|^\beta \Psi(|z|),$$

where $\Psi(x)$ is a slowly varying function, that is, for each fixed t $\lim_{x\to\infty} \frac{\Psi(tx)}{\Psi(x)} = 1$ (e.g., $\Psi(x) = \log^d x$ for some $d > 0$);
(O) *For all $z = \rho e^{i\theta}$ with $\theta \leq \pi$ such that $z \notin \mathcal{C}(D,\delta)$*

$$\rho = |z| > R \quad \Rightarrow \quad |\widetilde{G}(z)e^z| \leq A\rho^\gamma \exp[(1-\alpha\theta^2)\rho] \ ,$$

Then for every nonnegative integer m

$$\begin{aligned} g_n &= \sum_{i=0}^{m}\sum_{j=0}^{i+m} b_{ij} n^i \widetilde{G}^{\langle j\rangle}(n) + O(n^{\beta-(m+1)(2\delta-1)}\Psi(n)) \\ &= \widetilde{G}(n) + \sum_{k=1}^{m}\sum_{i=1}^{k} b_{i,k+i} n^i \widetilde{G}^{\langle k+i\rangle}(n) + O(n^{\beta-(m+1)(2\delta-1)}\Psi(n)), \end{aligned}$$

where b_{ij} are the coefficients defined in (10.40).

1.13 ⚠ Prove the central limit theorem for the Poisson model. More specifically, establish the following result.

Theorem 1.27 *Let X_N be a characteristic of the Poisson model with $\widetilde{G}(z,u) = \mathbf{E}u^{X_N}$, mean $\widetilde{X}(z)$ and variance $\widetilde{V}(z)$. Let the following hold for $z \to \infty$ in a cone $\mathcal{S}_\theta$ and for u belonging to a neighborhood $\mathcal{U}$ of $u = 1$ in the complex plane:*

$$\log \widetilde{G}(z,u) = O(A(z)),$$

where $A(z)$ is such that

$$\lim_{z\to\infty} \frac{A(z)}{\widetilde{V}^{3/2}(z)} = 0$$

and furthermore $\lim_{z\to\infty} \widetilde{V}(z) = \infty$. Then for complex τ

$$\widetilde{G}\left(z, e^{\tau/\sqrt{\widetilde{V}(z)}}\right) e^{-\tau \widetilde{X}(z)/\sqrt{\widetilde{V}(z)}} = e^{\frac{\tau^2}{2}}\left(1 + O\left(\frac{A(z)}{\widetilde{V}^{3/2}(z)}\right)\right).$$

Thus $\bar{X}_N = (X_N - \widetilde{X}(z))/\sqrt{\widetilde{V}(z)}$ converges in distribution and in moments to the standard normal distribution.

1.14 ⚠ (Jacquet and Szpankowski 1998) Prove the following extension of the depoissonization tool Theorem 10.13.

Theorem 1.28 *Let $\widetilde{G}_k(z)$ be a sequence of Poisson transforms of $g_{n,k}$, and each $\widetilde{G}_k(z)$ is assumed to be an entire function of z. Let $\log \widetilde{G}_k(z)$ exist in a cone $\mathcal{S}_\theta$. We suppose that there exists $\frac{1}{2} \le \beta < \frac{2}{3}$ such that the following two conditions hold:*

(I) For all $z \in \mathcal{S}_\theta$ such that for $|z| = n$

$$|\log \widetilde{G}_n(z)| \le Bn^\beta$$

for some constant $B > 0$.

(O) For all $z \notin \mathcal{S}_\theta$ such that $|z| = n$:

$$|\widetilde{G}_n(z)e^z| \le \exp(n - An^\alpha)$$

for some $\alpha > \beta$. Then for all $\varepsilon > 0$:

$$g_{n,n} = \widetilde{G}_n(n)\exp[-\frac{n}{2}(L_n'(n))^2]\left(1 + O(n^{3\beta-2+\varepsilon})\right),$$

where $L_n(z) = \log \widetilde{G}_n(z)$ and $L_n'(z) = \widetilde{G}_n'(z)/\widetilde{G}_n(z)$ is the first derivative of $L_n(z)$.

1.15 Consider again the leader election algorithm discussed in Section 10.5.1. In this exercise the reader is asked to consider the mean $\mathbf{E}[H_n]$ and the variance $\mathbf{Var}[H_n]$ of the height H_n. Prove the following result.

Theorem 1.29 *Define* $L := \ln 2$ *and* $\chi_k := 2\pi i k/L$.

(i) [**Prodinger, 1993**] *The average height* $\mathbf{E}[H_n]$ *satisfies*

$$\mathbf{E}[H_n] = \log_2 n + \frac{1}{2} - \delta_1(\log_2 n) + O\left(\frac{1}{n}\right),$$

where $\delta_1(\cdot)$ *is a periodic function of magnitude* $\leq 2 \times 10^{-5}$, *given by*

$$\delta_1(x) := \frac{1}{L} \sum_{\mathbb{Z}\setminus\{0\}} \zeta(1-\chi_k)\,\Gamma(1-\chi_k)\,e^{2\pi i k x},$$

and $\zeta(\cdot)$ *and* $\Gamma(\cdot)$ *denote Riemann's zeta function and Euler's gamma function, respectively.*

(ii) [**Fill, Mahmoud and Szpankowski, 1996**] *The variance* $\mathbf{Var}[H_n]$ *of the height satisfies*

$$\begin{aligned}\mathbf{Var}[H_n] &= \frac{\pi^2}{6\log^2 2} + \frac{1}{12} - \frac{2\gamma_1}{\log^2 2} - \frac{\gamma^2}{\log^2 2} + \delta_2(\log n) + O\left(\frac{\log n}{n}\right)\\ &= 3.116695\ldots + \delta_2(\log n) + O\left(\frac{\log n}{n}\right).\end{aligned}$$

Here the constants $(-1)^k\gamma_k/k!$, $k \geq 0$, *are the Stieltjes constants defined as*

$$\gamma_k := \lim_{m\to\infty}\left(\sum_{i=1}^{m}\frac{\log^k i}{i} - \frac{\log^{k+1} m}{k+1}\right).$$

In particular, $\gamma_0 = \gamma = 0.577215\ldots$ *is Euler's constant and* $\gamma_1 = -0.072815\ldots$. *The periodic function* $\delta_2(\cdot)$ *has magnitude* $\leq 2 \times 10^{-4}$.

1.16 (Janson and Szpankowski 1997) Extend the analysis of the leader election algorithm to biased coins, that is, assume that the probability of throwing a head is $p \neq \frac{1}{2}$.

1.17 Consider again the leader election algorithm discussed in Section 10.5.1 but this time the process stops when b leaders are selected. In other words, consider a b-incomplete trie version of the regular incomplete trie. Find the mean of the height (cf. Grabner [165]), the variance and the limiting distribution, if it exists.

1.18 ⚠ (Kirschenhofer, Prodinger and Szpankowski 1996) Consider the generalized probabilistic counting, as discussed in Example 2 of Chapter 9. The quantity $R_{n,d}$ determines the number of rounds before the algorithm terminates. Let $G_n(u) = \mathbf{E}u^{R_{n,d}}$, and $\widetilde{G}(z,u)$ be the probability generating function and its Poisson transform, respectively. Observe (cf. [252]) that

$$\widetilde{G}(z,u) = uf_d(z/2)\widetilde{G}(z/2,u) + (u-1)(f_d(z/2)-1) \ ,$$

where $f_d(z) = 1 - e_d(z)e^{-z}$ and $e_d(z) = 1 + \frac{z^1}{1!} + \cdots + \frac{z^d}{d!}$ is the truncated exponential function. Then prove the following result.

Theorem 1.30 (Kirschenhofer, Prodinger and Szpankowski, 1996) *For any integer m*

$$\Pr\{R_{n,d} \le \log_2 n + m - 1\} = 1 - \varphi\left(2^{-m-\langle \log_2 n\rangle}\right) + O(n^{-1}),$$

where

$$\varphi(z) = \prod_{j=0}^{\infty} f_d(z2^j) = \prod_{j=0}^{\infty}\left(1 - e_d(z2^j)e^{-z2^j}\right)$$

and $\langle \log n\rangle = \log n - \lfloor \log n\rfloor$.

1.19 Let

$$h(t) = -p^{-t}\log p - q^{-t}\log q$$

and s_k be a root of $1-p^s-q^{-s}=0$ for $k\in\mathbb{Z}$. Prove that if $\Re(s_k) = -1$, then $h(s_k) = h$, where h is the entropy of *Bernoulli*(p).

1.20 Prove that the constant A defined in (10.91) simplifies to $A = H_b + b(1+b)^{-1}$.

1.21 ⚠ (Louchard, Szpankowski and Tang 1999) Consider again the b-digital search tree model in Section 10.5.2 and prove the following limiting distribution for the depth D_m.

Theorem 1.31 *Let $G_m(u)$ be the probability generating function of D_m (i.e., $G_m(u) = \mathbf{E}[u^{D_m}]$), $\mu_m = ED_m$, and $\sigma_m = \sqrt{\mathbf{Var}\ D_m}$. Then for complex τ*

$$e^{-\tau\mu_m/\sigma_m}G_m(e^{\tau/\sigma_m}) = e^{\frac{\tau^2}{2}}\left(1 + O\left(\frac{1}{\sqrt{\log m}}\right)\right) \ .$$

Thus the limiting distribution of $\frac{D_m-\mu_m}{\sigma_m}$ is normal, and it converges in moments to the appropriate moments of the standard normal distribution. Also, there exist positive *constants A and $\alpha < 1$ (that may depend on p and b) such that uniformly in k, and for large m*

$$\Pr\left\{\left|\frac{D_m - c_1 \log m}{\sqrt{c_2 \log m}}\right| > k\right\} \leq A\alpha^k,$$

where $c_1 = 1/h$ and $c_2 = (h_2 - h^2)/h^3$.

1.22 Prove (10.95) and (10.96).

1.23 Prove Lemma 10.24.

1.24 ⚠ (Jacquet, Szpankowski and Tang 2000) Consider a digital search tree under the Markovian model (M) as in Section 10.5.3 and prove the following limiting distribution for the depth D_m

$$\frac{D_m - \mathbf{E}[D_m]}{\sqrt{\mathbf{Var} D_m}} \to N(0,1),$$

where $N(0,1)$ represents the standard normal distribution and $\mathbf{E}[D_m]$ and $\mathbf{Var}[D_m]$ are computed in Theorem 10.23.

1.25 ⚠ (Jacquet, Szpankowski and Tang 2000) Extend Theorem 10.23 to a *non-stationary* Markov model.

1.26 Extend Theorem 10.23 to b-DST under the Markovian model.

1.27 ⚠ (Jacquet, Szpankowski and Tang 2000) Using the analysis from Section 10.5.3, establish the mean, the variance and the limiting distribution for the phrase length in the Lempel-Ziv'78 scheme, when a string of length n is generated by a Markovian source.

1.28 ∇? Consider the digital search tree model in Section 10.5.3 with $b = 1$ and where strings are generated by m independent Markov sources. Let L_m be the total path length, that is, the sum of all depths. Prove that $E[L_m] = m\mathbf{E}[D_m]$ and $\mathbf{Var}[L_m] \sim m\mathbf{Var}[D_m]$, where $\mathbf{E}[D_m]$ and $\mathbf{Var}[D_m]$ were computed in Theorem 10.23. Then show that $(L_m - \mathbf{E}[D_m])/\mathbf{Var}[D_m]$ tends to the standard normal distribution. Finally, prove similar results for the path lengths for regular tries and PATRICIA tries.

1.29 ∇? Consider the Lempel-Ziv'78 scheme, where a string of length n is generated by a Markovian source. Use the same notation as in Section 10.5.3. In particular, let

$$c_2 = \frac{\ddot{\lambda}(-1) - \dot{\lambda}^2(-1)}{\dot{\lambda}^3(-1)},$$

where $\lambda(s)$ is the principal eigenvalue of $\mathsf{I} - \mathsf{P}(s)$. Prove that the number of phrases $(M_n - \mathbf{E}[M_n])/\mathbf{Var}[M_n]$ tends to the standard normal distribution with

$$\mathbf{E}[M_n] \sim \frac{nh}{\log n},$$
$$\mathbf{Var}[M_n] \sim \frac{c_2 h^3 n}{\log^2 n},$$

where h is the entropy rate of the Markov source. This is an open and a *very* difficult problem.

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