Chapter 9

Mellin Transform and Its Applications

The Mellin transform is the most popular transform in the analysis of algorithms. It is closely related to the two-sided Laplace and Fourier transforms except that it has a polynomial kernel. D. E. Knuth, together with De Bruijn, introduced it in the orbit of discrete mathematics in the mid 60's, however, Flajolet's school systematized and applied the Mellin transform to a myriad problems of analytic combinatorics and analysis of algorithms. Recently, the Mellin transform found its way into information theory. The popularity of this transform stems from two important properties. It allows the reduction of certain functional equations to algebraic ones, and it provides a direct mapping between asymptotic expansions of a function near zero or infinity and the set of singularities of the transform in the complex plane. The latter asymptotic property, enriched in the singularity analysis or depoissonization (discussed in the next chapter), is crucial for applications in discrete mathematics, analysis of algorithms, analytic combinatorics and analytic information theory. We discuss here some properties of the Mellin transform, and illustrate its applications with many examples.

THE MELLIN TRANSFORM $f^*(s)$ of a complex-valued function $f(x)$ defined over positive reals is

$$\mathcal{M}[f(x);s] := f^*(s) = \int_0^\infty f(x)x^{s-1}dx$$

with s being a complex number. The Mellin transform can be viewed as the Laplace transform (set $x = e^t$) or the Fourier transform (set $x = e^{i\omega}$). However, it proves convenient to work with the Mellin transform rather than the Laplace-Fourier version. This is particularly true in the analysis of algorithm and analytic combinatorics where one often deals with functional equations like

$$f(x) = a(x) + \alpha f(xp) + \beta f(xq)$$

where α, β are constants, and $a(x)$ is a known function. We have already encountered such functional equations in this book (e.g., Sections 7.6.1 and 8.7.3). We will see more of these in the forthcoming chapter. The point is that the Mellin transform maps the above functional equation into an algebraic one that is easier to solve and hence allows us to recover $f(x)$, at least asymptotically as $x \to 0$ or $x \to \infty$.

The usefulness of the Mellin transform stems for its asymptotic properties. There is a direct mapping between asymptotic expansions of a function near zero or infinity and the set of singularities of the transform in the complex plane. This plays a crucial role in applications.

For example, sums like

$$G(x) = \sum_{k=0}^{\infty} \left(1 - e^{-x/2^k}\right) \qquad \text{and} \qquad H(x) = \sum_{k=1}^{\infty} (-1)^k e^{-k^2 x} \log k$$

that are not easily computable, either numerically of asymptotically, can be treated by Mellin transforms for $x \to 0$ or $x \to \infty$. We study some representative examples in this chapter.

In the analysis of algorithms, $f(x)$ is often a generating function of a sequence f_n. Using the asymptotic properties of the Mellin transform, we are able to find an asymptotic expansion of $f(x)$, but we still need a tool to recover f_n (this is called a *two-step approach*). We may either use the singularity analysis discussed in the last chapter or turn to another approach called *analytic depoissonization* that we shall introduce in this next chapter.

Over the last thirty years, a myriad of analyses of algorithms have been successfully accomplished through Mellin transforms. The list is quite long, and we mention here only: sorting and searching methods [220], digital trees such as tries [172, 173, 175, 199, 220, 206, 320, 319], PATRICIA trie [207, 220, 274, 321], digital search trees [116, 177, 180, 209, 220, 322] and suffix trees [176], string matching [26, 262], data compression [244, 177, 180], multidimensional searching [107], communication protocols [137, 250, 317], randomized data structures [203], and probabilistic counting [210]. In some cases, notably digital search trees and sorting, alternative methods (e.g., Rice method) can be used. However, when the problem becomes more complicated, it is fair to say that the Mellin transform must be involved. We shall see it in this chapter, and even more in the next one.

We start this chapter with a short discussion of basic properties of the Mellin transform followed by a longer "excursion" into asymptotic properties. We also present a brief extension of the Mellin transform to complex plane that is crucial for applications. As always, we finish with an application section where we discuss the average and the variance of the depth in an extension of digital search trees, and then we evaluate the minimax redundancy of a renewal process. The latter example is one of the most involved in this book.

We intend to make this chapter short since there are excellent and in–depth surveys on the Mellin transform. The recent extended survey by Flajolet, Gourdon and Dumas [115] contains more than we plan to discuss here. We shall borrow from it freely, and refer the reader to it for further details. An even more detailed account on the method can be found in the forthcoming book of Flajolet and Sedgewick [118]. Brief discussions are also in the books by Hofri [160] and Mahmoud [249].

9.1 Basic Properties

Let $f(x)$ be a complex-valued function that exists over $(0, \infty)$ and is locally integrable. To avoid further complication, we assume throughout that f is continuous in $(0, \infty)$. The Mellin transform is defined as

$$\mathcal{M}[f(x); s] := f^*(s) = \int_0^{\infty} f(x) x^{s-1} dx. \tag{9.1}$$

In the sequel, we study the existence of the transform, compute the transform of some common functions, and investigate several functional properties of the transform.

(M1) Fundamental Strip
Let $f(x)$ be a continuous function on the interval $(0,\infty)$ such that

$$f(x)=\begin{cases} O(x^{\alpha}) & x\to 0 \\ O(x^{\beta}) & x\to\infty\ . \end{cases}$$

Then the Mellin transform $f^*(s)$ exists for any complex number s in the *fundamental strip* $-\alpha<\Re(s)<-\beta$, which we also denote as $\langle -\alpha,-\beta\rangle$. This is true since

$$\begin{aligned}\left|\int_0^\infty f(x)x^{s-1}dx\right| &\le \int_0^1 |f(x)|x^{\Re(s)-1}dx+\int_1^\infty |f(x)|x^{\Re(s)-1}dx \\ &\le c_1\int_0^1 x^{\Re(s)+\alpha-1}dx+c_2\int_1^\infty x^{\Re(s)+\beta-1}dx\end{aligned}$$

where c_1 and c_2 are constants. The first integral above exists for $\Re(s)>-\alpha$ while the second for $\Re(s)<-\beta$. This proves the claim. For example, $f_1(x)=e^{-x}$, $f_2(x)=e^{-x}-1$, $f_3(x)=1/(1+x)$ have the fundamental strips $\langle 0,\infty\rangle$, $\langle -1,0\rangle$ and $\langle 0,1\rangle$, respectively (see also Table 9.1). In passing, we observe that the Mellin transform is *analytic* inside its fundamental strip. We also notice that the Mellin transform of a polynomial does not exist. However, as we shall see, polynomials play a crucial role; they shift the fundamental strip (cf. Entries 1,2, and 3 in Table 9.1).

Table 9.1 presents some of the most commonly used Mellin transforms with their corresponding fundamental strips. These formulas will be established in the course of the next few pages. For now we observe that Entry 1 of the table is the classical Euler's gamma function. Entries 5 and 7 follow from the classical beta function. Indeed,

$$f^*(s)=\int_0^\infty x^{s-1}(1+x)^{-1}dx=\int_0^1 t^s(1-t)^{1-s}dt=\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin\pi s}$$

where the last equation is a consequence of (2.32). Entries 8-11 are trivial to prove, but they play the most important role in the asymptotic analysis of the Mellin transform. As we shall see in Section 9.2, these formulas remain valid even when $=$ is replaced by $\sim$ for $x\to 0$ and $x\to\infty$, respectively.

(M2) Smallness of Mellin transforms
Let $s=\sigma+it$. We first only assume that $f(x)$ is continuous. Observe that

$$f^*(\sigma+it)=\int_0^\infty f(x)e^{-\sigma x}e^{it\log x}dx,$$

Table 9.1: Mellin Transform of Some Functions

Item	Function	Mellin Transform	Fund. Strip
1.	e^{-x}	$\Gamma(s)$	$\langle 0, \infty \rangle$
2.	$e^{-x} - 1$	$\Gamma(s)$	$\langle -1, 0 \rangle$
3.	$e^{-x} - 1 + x$	$\Gamma(s)$	$\langle -2, -1 \rangle$
4.	e^{-x^2}	$\frac{1}{2}\Gamma\left(\frac{s}{2}\right)$	$\langle 0, \infty \rangle$
5.	$\frac{1}{1+x}$	$\frac{\pi}{\sin \pi s}$	$\langle 0, 1 \rangle$
6.	$\log(1+x)$	$\frac{\pi}{s \sin \pi s}$	$\langle -1, 0 \rangle$
7.	$\frac{1}{(1+x)^\nu}$	$B(s, \nu - s)$	$\langle 0, \nu \rangle, \quad \nu > 0$
8.	$H(x) = I(0 \le x \le 1)$	$\frac{1}{s}$	$\langle 0, \infty \rangle$
9.	$1 - H(x)$	$-\frac{1}{s}$	$\langle -\infty\, 0 \rangle$
10.	$x^a \log^k(x) H(x)$	$\frac{(-1)^k k!}{(s+a)^{k+1}}$	$\langle -a, \infty \rangle, \quad k \in \mathbb{N}$
11.	$x^a \log^k(x)(1 - H(x))$	$-\frac{(-1)^k k!}{(s+a)^{k+1}}$	$\langle -\alpha, \infty \rangle, \quad k \in \mathbb{N}$
12.	$\frac{e^{-x}}{1-e^{-x}}$	$\Gamma(s)\zeta(s)$	$\langle 1, \infty \rangle$
13.	$\log\left(\frac{1-e^{-x}}{x}\right)$	$-\frac{\Gamma(s+1)\zeta(s+1)}{s}$	$\langle -1, 0 \rangle$
14.	$\log(1 - e^{-x})$	$-\Gamma(s)\zeta(s+1)$	$\langle 0, \infty \rangle$

thus by the Riemann-Lebesgue Lemma 8.14

$$f^*(\sigma + it) = o(1) \qquad \text{as} \quad t \to \pm\infty. \tag{9.2}$$

If $f(x)$ is r times differentiable, then by integration by parts we have $\mathcal{M}[f^{(r)}(x); s] = (-1)^r (s - r)^r f^*(s - r)$ (cf. also (9.8) below). This, together with (9.2), yields

$$f^*(\sigma + it) = o(|t|^{-r}) \qquad \text{as} \quad t \to \pm\infty \tag{9.3}$$

for σ fixed.

(M3) Functional Properties

The following properties play a prime role in applications of the Mellin transform. They are easy to establish in the corresponding fundamental strips:

Function	$\Leftrightarrow$	**Mellin Transform**		
$x^{\nu} f(x)$	$\Leftrightarrow$	$f^*(s+\nu)$,		(9.4)
$f(\mu x)$	$\Leftrightarrow$	$\mu^{-s} f^*(s)$	$\mu > 0$,	(9.5)
$f(x^{\rho})$	$\Leftrightarrow$	$\frac{1}{\rho} f^*(s/\rho)$	$\rho > 0$,	(9.6)
$f(x)\log x$	$\Leftrightarrow$	$\frac{d}{ds} f^*(s)$,		(9.7)
$\frac{d}{dx} f(x)$	$\Leftrightarrow$	$-(s-1) f^*(s-1)$		(9.8)
$x\frac{d}{dx} f(x)$	$\Leftrightarrow$	$-s f^*(s)$		(9.9)
$\int_0^x f(t) dt$	$\Leftrightarrow$	$-\frac{1}{s} f^*(s+1)$		(9.10)
$f(x) = \sum_{k\in\mathcal{K}} \lambda_k g(\mu_k x)$	$\Leftrightarrow$	$f^*(s) = g^*(s) \sum_{k\in\mathcal{K}} \lambda_k \mu_k^{-s}$,	$\mathcal{K}$ finite	(9.11)

Formulas (9.4)-(9.11) are direct and simple consequences of the Mellin transform definition. For example, (9.9) follows from integration by parts

$$\mathcal{M}[x\frac{d}{dx} f(x); s] = f(x)x^s|_0^{\infty} - s\mathcal{M}[f(x); s] = -s\mathcal{M}[f(x); s]$$

with the validity in a strip, that is a subset of the fundamental strip, dictated by the growth of f.

Using the above properties, we may establish some more entries from Table 9.1. In particular, Entry 4 follows directly from Entry 1 and (9.6). To prove Entry 6 we apply property (9.10) to Entry 5. To establish Entries 12-14, we need to generalize property (9.11) to infinite $\mathcal{K}$, which we discuss next.

The sum

$$F(x) = \sum_k \lambda_k f(\mu_k x) \tag{9.12}$$

is called the *harmonic sum.* After formally applying the scaling property (9.5), and assuming the corresponding series exist, we obtain

$$F^*(s) = f^*(s) \sum_k \frac{\lambda_k}{\mu_k^s}. \tag{9.13}$$

This formula is responsible, to a large extent, for usefulness of the Mellin transform and finds plenty of applications in discrete mathematics, analysis of algorithms and analytic combinatorics that we discuss in the sequel. In particular, it proves Entries 12-14 of Table 9.1. The theorem below provides conditions under which it is true.

Theorem 9.1 (Harmonic Sum Formula) *The Mellin transform of the harmonic sum (9.12) is defined in the intersection of the fundamental strip of $f(x)$ and the domain of absolute convergence of the Dirichlet series $\sum_k \lambda_k \mu_k^{-s}$ (which is of the form $\Re(s) > \sigma_a$ for some real σ_a), and it is given by (9.13).*

Proof. Since $f^*(s)$ and the Dirichlet series $\sum_k \lambda_k \mu_k^{-s}$ are both analytic in the intersection, if not empty, of the corresponding convergence regions, the interchange of summation and integration is legitimate by Fubini's theorem. ∎

To prove Entry 12 of Table 9.1 we proceed as follows. For any $x > 0$ we have

$$\frac{e^{-x}}{1-e^{-x}} = \sum_{k=1}^{\infty} e^{-kx}.$$

By (9.13) its Mellin is $\mathcal{M}[e^{-x}; s] \sum_{k=1}^{\infty} k^{-s} = \Gamma(s)\zeta(s)$ for $\Re(s) > 1$. For Entry 14 we need to wrestle a little more, but

$$\mathcal{M}[\log(1-e^{-x}); s] = \mathcal{M}\left[\int_0^x \frac{e^{-t}}{1-e^{-t}} dt; s\right] = -\frac{1}{s}\Gamma(s+1)\zeta(s+1) = -\Gamma(s)\zeta(s+1).$$

As a consequence of this, we immediately prove that

$$\mathcal{M}\left[\sum_{k\geq 1} \log(1-e^{-kx}); s\right] = -\Gamma(s)\zeta(s)\zeta(s+1) \tag{9.14}$$

for $\Re(s) > 1$. The reader is asked to prove Entry 13 in Exercise 1.

(M4) Inverse Mellin Transform

Since the Mellin transform is a special case of the Fourier transform, we should expect a similar inverse formula. Indeed this is the case. Let $s = \sigma + it$ and set $x = e^{-y}$. Then

$$f^*(s) = -\int_{-\infty}^{\infty} f(e^{-y}) e^{-\sigma y} e^{-ity} dy.$$

This links Mellin and Fourier transforms, and from the inverse Fourier transform we obtain (cf. Henrici [158]): *if $f(x)$ is continuous on $(0, \infty)$, then*

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(s) x^{-s} ds \tag{9.15}$$

where $a < c < b$ and $\langle a, b\rangle$ is the fundamental strip of $f^*(s)$. We should remind the reader that we have already encountered an integral like the one in (9.15) in Section 7.6.3 and Section 8.5 (e.g., Theorem 8.21 and the Rice's method).

The inverse formula is of prime importance to asymptotics. We first prove Entry 2 of Table 9.1. We apply (9.15) with $\Gamma(s)$ defined in the strip $\langle -1, 0\rangle$. Let $c = -\frac{1}{2}$ and consider a large rectangle *left* to the line of integration in (9.15) with the left line located at, say $-M - \frac{1}{2}$ for $M > 0$. As in the proof of Theorem 8.21 we show that the top and the bottom line contribute negligible while the left line contributes $O(x^{M+1/2})$. Thus

$$\begin{aligned} f(x) &= \frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(s)x^{-s}ds = \sum_{k=1}^{M} \mathrm{Res}[\Gamma(s); s=-k] + O(x^{M+1/2}) \\ &= \sum_{k=1}^{M} \frac{(-1)^k}{k!}x^k + O(x^{M+1/2}) \\ &\to e^{-x} - 1 \qquad \text{as} \quad M \to \infty \end{aligned}$$

that proves Entry 2. In a similar manner we can prove Entry 3 (cf. also Exercise 2). This completes the derivations of all entries in Table 9.1.

In the introduction we mentioned that Mellin transforms are useful to solve functional equations. We shall illustrate it in Example 2 below, while in the next example we show that the Mellin transform can also be used to establish functional equations.

Example 9.1 *A Functional Equation Through the Mellin Transform*

In the analysis of the variance of a certain parameter (i.e., partial match) in a multidimensional search Kirschenhofer *et al.* [208] studied the following function

$$F(x) = \sum_{k\ge 1} \frac{e^{-kx}}{1+e^{-2kx}}.$$

As a matter of fact, it was already analyzed by Ramanujan (cf. [35]). Our goal is to prove for following function equation on $F(x)$

$$F(x) = \frac{\pi}{4x} - \frac{1}{4} + \frac{\pi}{x}F\left(\frac{\pi^2}{x}\right). \tag{9.16}$$

Let

$$\beta(s) = \sum_{j\ge 0}(-1)^j \frac{1}{(2j+1)^s}$$

that resembles the zeta function. Observe that

$$F(x) = \sum_{k\ge 1}\frac{e^{-kx}}{1+e^{-2kx}} = \sum_{j\ge 0}(-1)^j \sum_{k\ge 1} e^{-k(2j+1)x}.$$

Then, by the harmonic sum formula, the Mellin transform $F^*(s)$ of $F(x)$ is

$$F^*(s) = \Gamma(s)\zeta(s)\beta(s),$$

By the Mellin inversion formula this yields

$$F(x) = \frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty} \Gamma(s)\zeta(s)\beta(s)x^{-s}ds\ .$$

Now we take the two residues at $s = 1$ and $s = 0$ out from the above integral (notice that $\beta(0) = 1/2$ and $\beta(1) = \pi/4$; cf. [2]) and apply the duplication formula for $\Gamma(s)$ (cf. Section 2.4.1) to obtain

$$F(x) = \frac{\pi}{4x} - \frac{1}{4} + \frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{1}{\sqrt{\pi}}2^{s-1}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)x^{-s}\zeta(s)\beta(s)ds\ .$$

As before, the exponential smallness of the Γ-function along vertical lines justifies the shifting of the line integral. We now use the functional equations for $\zeta(s)$ and $\beta(s)$ (cf. Section 2.4), namely

$$\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{s-\frac{1}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \tag{9.17}$$

and

$$\beta(1-s)\Gamma\left(1-\frac{s}{2}\right) = 2^{2s-1}\pi^{-s+\frac{1}{2}}\Gamma\left(\frac{s+1}{2}\right)\beta(s)\ . \tag{9.18}$$

Identity (9.17) is Riemann's functional equation for $\zeta(s)$, and (9.18) is an immediate consequence of the functional equation for Hurwitz's ζ-function $\zeta(s, a)$ (cf. [15]), and the fact that

$$\beta(s) = 4^{-s}\left[\zeta(s, \frac{1}{4}) - \zeta(s, \frac{3}{4})\right]\ .$$

Using (9.17) and (9.18), and substituting $1 - s = u$, we arrive at

$$F(x) = \frac{\pi}{4x} - \frac{1}{4} + \frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \pi^{1-2u}\Gamma(u)x^{u-1}\zeta(u)\beta(u)du\ .$$

This proves (9.16).

In passing we mention that the functional equation can be used for a numerical evaluation of $F(x)$, as already observed in [205]. For example, in applications the following series is of interest of

$$F(\log 2) = \sum_{k\geq 1}\frac{2^k}{2^{2k}+1}.$$

Since $\pi^2/\log 2 \approx 14.2$ we expect $F(\pi^2/\log 2)$ to be small so that the first two terms in (9.16) will determine the numerical value of $F(\log 2)$. As a matter of fact, pulling out one term from $F(\pi^2/\log 2)$ and computing other terms we obtain $F(\log 2) = 0.8830930036(10^{-11})$.

In Exercise 4 we ask the reader to prove similar functional equations. □

9.2 Asymptotic Properties of the Mellin Transform

We said it many times, and we repeat it again, usefulness of the Mellin transform stems from its asymptotic properties that we shall discuss in this section. We start with an informal discussion that will lead to the direct and reverse mapping theorems, the pillars of the Mellin transform asymptotics.

Let $f(x)$ be r times differentiable function with $r \geq 2$. The fundamental strip of $f^*(s)$ is assumed to be $\alpha < \Re(s) < \beta$. We restrict analysis to meromorphic functions $f^*(s)$ that can be analytically continued to $\beta \leq \Re(s) \leq M$ for some M. We also postulate that $f^*(s)$ has finitely many poles λ_k such that $\Re(\lambda_k) < M$. Then we claim that for $x \to \infty$

$$f(x) = -\sum_{\lambda_k \in \mathcal{K}} \mathrm{Res}[f^*(s)x^{-s}, s = \lambda_k] + O(x^{-M}) \qquad x \to \infty \tag{9.19}$$

where $\mathcal{K}$ is the set of singularities and M is as large as we want. (In a similar fashion one can continue the function $f^*(s)$ to the left to get an asymptotic formula for $x \to 0$.) This is the announced asymptotic property. Its proof is quite simple and we have already seen the main ingredients of it in Section 7.5.3. Due to its importance, we repeat below the main steps of the analysis.

Consider the rectangle $\mathcal{R}$ given in Figure 9.1 with the corners as illustrated. Choose A so that the sides of $\mathcal{R}$ do not pass through any singularities of $f^*(s)x^{-s}$. When evaluating

$$\lim_{A\to\infty} \int_R = \lim_{A\to\infty} \left(\int_{c-iA}^{c+iA} + \int_{c+iA}^{M+iA} + \int_{M+iA}^{M-iA} + \int_{M-iA}^{c-iA} \right),$$

the second and fourth integrals contribute $O(A^{-r})$ due to the smallness property (M2). The contribution of the fourth integral is computed as follows:

$$\begin{aligned} \left| \int_{M+i\infty}^{M-i\infty} f^*(s)x^{-s} ds \right| &= \left| \int_{\infty}^{-\infty} f^*(M+it)x^{-M-it} dt \right| \\ &\leq |x^{-M}| \int_{\infty}^{-\infty} |f^*(M+it)||x^{-it}| dt = O(x^{-M}) \end{aligned}$$

since f is at least twice differentiable and thus the integral above exists. Using now the Cauchy residue theorem and taking into account the negative direction of $\mathcal{R}$ we finally obtain

$$-\sum_{\lambda_k \in \mathcal{H}} \mathrm{Res}[f^*(s)x^{-s}, s = \lambda_k] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(s)x^{-s} ds + O(x^{-M}),$$

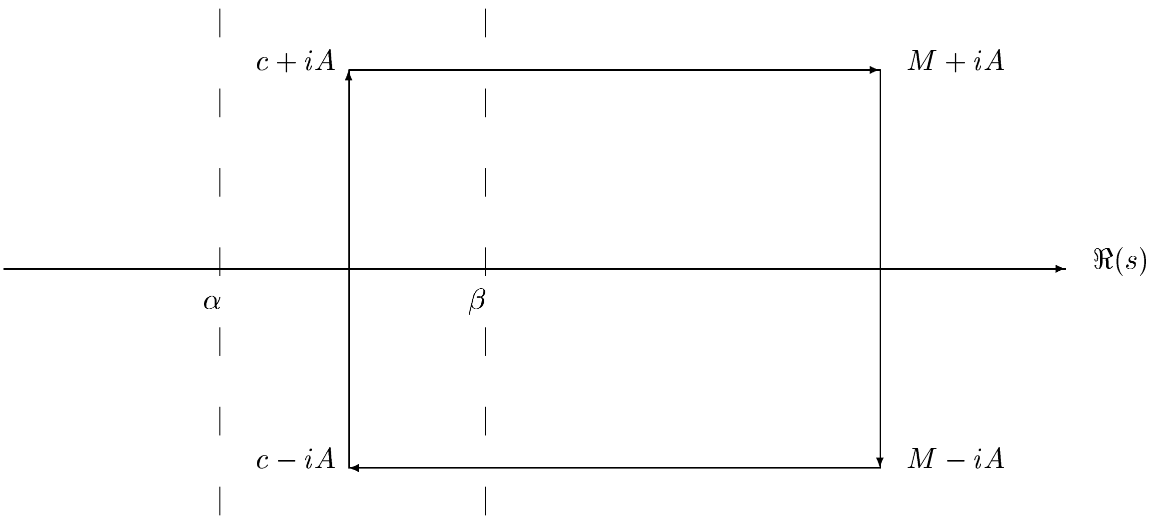

Figure 9.1: The fundamental strip of $f^*(s)$ and the integration contour

where $\alpha < c < \beta$. This proves (9.19).

In particular, if $f^*(s)$ has the following form

$$f^*(s) = \sum_{k=0}^{K} \frac{d_k}{(s-b)^{k+1}} \,, \tag{9.20}$$

and $f^*(s)$ can be analytically continued to $\beta < \Re(s) \le M$, then Entry 10 of Table 9.1 implies that

$$f(x) = -\sum_{k=0}^{K} (-1)^k \frac{d_k}{k!} x^{-b} \log^k x + O(x^{-M}) \qquad x \to \infty. \tag{9.21}$$

In a similar fashion, if for $-M < \Re(s) < \alpha$ the smallness condition of $f^*(s)$ holds and

$$f^*(s) = \sum_{k=0}^{K} \frac{d_k}{(s-b)^{k+1}} \,, \tag{9.22}$$

then

$$f(x) = \sum_{k=0}^{K} (-1)^k \frac{d_k}{k!} x^{-b} \log^k x + O(x^{M}) \qquad x \to 0. \tag{9.23}$$

We return to these relationships below when we establish the direct and reverse mapping theorems.

Example 9.2 *Splitting Process Arising in Probabilistic Counting*
Splitting processes arise in many applications that include probabilistic counting [210], selecting the loser [100, 272], estimating the number of questions necessary to identify distinct objects [269], searching algorithms based on digital tries [172, 173, 220, 249, 319] approximate counting [200], conflict resolution algorithms for multiaccess communication [174, 250, 318] and so forth. Let us focus on the probabilistic counting. Using a digital tree representation one can describe it in terms of this splitting process as follows: Imagine n persons flipping a fair coin, and those who get 1 discontinue throwing (move to the right in the digital tree representation) while the rest continue throwing (i.e., move to the left in the digital tree). The process continues until all remaining persons flip a 0. Consider now the following *more general version* of the splitting process, where the coin flipping ends as soon as at most d persons have flipped a 1 in the same round, where d is a given parameter. We denote the number of rounds by $1 + R_{n,d}$. Observe that $d = 0$ corresponds to the original situation.

Kirschenhofer *et al.* [210] proved that the so called Poisson transform (see also next chapter)

$$\widetilde{L}(z) = e^{-z} \sum_{n \geq 0} \mathbf{E}[R_{n,d}] \frac{z^n}{n!}$$

satisfies the following functional equation

$$\widetilde{L}(z) = f_d(z/2)\widetilde{L}(z/2) + f_d(z/2)$$

where $f_d(z) = 1 - e_d(z)e^{-z}$ and

$$e_d(z) = 1 + \frac{z^1}{1!} + \cdots + \frac{z^d}{d!}.$$

After iterating this equation we arrive at (cf. Section 7.6.1)

$$\widetilde{L}(z)\varphi(z) = \sum_{n=0}^{\infty} \varphi(z2^{-n-1}) \ , \tag{9.24}$$

where

$$\varphi(z) = \prod_{j=0}^{\infty} f_d(z2^j) = \prod_{j=0}^{\infty} \left(1 - e_d(z2^j)e^{-z2^j}\right) \ . \tag{9.25}$$

By d'Alembert's criterion, the final expression converges absolutely for all complex numbers z. We derive an asymptotic expansion of $\widetilde{L}(z)$ for $z \to \infty$ (and for now assume that z is real).

Observe that (9.24) is a harmonic sum. Let $Q_1(z) = \widetilde{L}(z)\varphi(z)$ and $Q_1^*(s)$ be its Mellin transform that exists in the strip $\Re(s) \in (-\infty, 0)$. By the harmonic sum formula we have

$Q_1^*(s) = 2^s/(1-2^s)\varphi^*(s)$ where the Mellin transform $\varphi^*(s)$ of $\varphi(x)$ exists in $\Re(s) \in (-\infty, 0)$. However, there is a problem since the Mellin transform of $\varphi(s)$ does not exists at $s = 0$, and we need a more precise estimate of $\varphi^*(s)$ at $s = 0$. Thus, we proceed as follows. Define

$$\Phi(x) = \varphi(2x) - \varphi(x) = e_d(x)e^{-x}\varphi(2x) \ .$$

Observe now that the Mellin transform of $\Phi(x)$ exists in $(-\infty, \infty)$, and

$$Q_1^*(s) = \frac{2^s}{1-2^s}\varphi^*(s) = \left(\frac{2^s}{1-2^s}\right)^2 \Phi^*(s) \ , \tag{9.26}$$

where $\Phi^*(s)$ is an entire function, as we noticed above, defined as

$$\Phi^*(s) = \int_0^\infty e_d(x)e^{-x}\varphi(2x)x^{s-1}dx \ .$$

To evaluate $\widetilde{L}(z)$ asymptotically we apply (9.19). Note that $\chi_k = 2\pi i k/\log 2$ ($k = 0, \pm 1, \pm 2, \ldots$) are the solutions of $1 - 2^{-s} = 0$, and the main contribution to the asymptotics comes from $\chi_0 = 0$. Since $\varphi(z) \sim 1 + O(z^{-M})$ for any $M > 0$, we obtain, for $z \to \infty$

$$\widetilde{L}(z) = \log_2 z \cdot \frac{\Phi^*(0)}{\log 2} - \frac{\Phi^*(0)}{\log 2} - \frac{\Phi^{*\prime}(0)}{\log^2 2} + P_1(\log_2 z) + O(z^{-M})$$

where

$$\begin{aligned} \Phi^*(0) &= \int_0^\infty e^{-x}e_d(x)\varphi(2x)\frac{dx}{x}, \\ \Phi^{*\prime}(0) &= \int_0^\infty e^{-x}e_d(x)\varphi(2x)\frac{\log x}{x}dx, \end{aligned}$$

and

$$P_1(x) = \sum_{k\neq 0}\left(\frac{\Phi^*(-\chi_k)(x-1)}{\log 2} - \frac{\Phi^{*\prime}(-\chi_k)}{\log^2 2}\right)e^{2k\pi i x} \ .$$

To complete the proof, we need to evaluate $\Phi^*(0)$,$\Phi^*(\chi_k)$, $\Phi^{*\prime}(0)$, and $\Phi^{*\prime}(\chi_k)$. We show here how to estimate $\Phi^*(0)$, the other evaluations can be found in [210]. Define the function $\mathbf{1}(x)$ as follows

$$\mathbf{1}(x) = \begin{cases} 1 & \text{if } x \geq 1 \\ 0 & \text{if } x < 1 \ . \end{cases}$$

Then, we can write

$$\begin{aligned} \Phi^*(0) &= \int_0^\infty (\varphi(2x) - \varphi(x))\frac{dx}{x} \\ &= \int_0^\infty (\varphi(2x) - \mathbf{1}(2x))\frac{dx}{x} + \int_0^\infty (\mathbf{1}(2x) - \mathbf{1}(x))\frac{dx}{x} + \int_0^\infty (\mathbf{1}(x) - \varphi(x))\frac{dx}{x} \\ &= \int_0^\infty (\mathbf{1}(2x) - \mathbf{1}(x))\frac{dx}{x} = \int_{1/2}^1 \frac{dx}{x} = \log 2, \end{aligned}$$

since after the substitution $u = 2x$ in the second line of the above display, the first and the second integrals cancel. A similar derivation shows that $\Phi^*(\chi_k) = 0$. $\square$

We are now in a position to state precisely results regarding asymptotic properties of the Mellin transform. We start with the so called *reverse mapping* that maps the asymptotics of the Mellin transform into asymptotics of the original function for $x \to 0$ and $x \to \infty$. Following Flajolet, Gourdon and Dumas [115] we introduce the *singular expansion*. It is basically a sum of the Laurent expansions around *all* poles truncated to the $O(1)$ term. For example, since

$$\begin{aligned} \frac{1}{s(s-1)} &= \frac{-1}{s} - 1 + O(s), \qquad s \to 0, \\ \frac{1}{s(s-1)} &= \frac{1}{s-1} - 1 + O(s-1), \qquad s \to 1, \end{aligned}$$

we write

$$\frac{1}{s(s-1)} \asymp \left[\frac{-1}{s} - 1\right]_{s=0} + \left[\frac{1}{s-1} - 1\right]_{s=1}$$

as a singular expansion. In general, for a meromorphic function $f(s)$ with poles in $\mathcal{K}$ the singular expansion is

$$f(s) \asymp \sum_{w \in \mathcal{K}} \Delta_w(s)$$

where $\Delta_w(s)$ is the Laurent expansion of f around $s = w$ up to at most $O(1)$ term.

Based on our discussion above we formulate the following result.

Theorem 9.2 (Reverse Mapping) *Let f be continuous with Mellin transform $f^*(s)$ defined in a non-empty strip $\langle \alpha, \beta \rangle$.*

(i) [ASYMPTOTICS FOR $x \to \infty$] *Assume that $f^*(s)$ admits a meromorphic continuation to $\Re(s) \in (\alpha, M)$ for some $M > \beta$ such that*

$$f^*(s) = O(|s|^{-r}), \qquad s \to \infty \tag{9.27}$$

for some $r > 1$ and

$$f^*(s) \asymp \sum_{k,b} \frac{c_{k,b}}{(s-b)^{k+1}} \tag{9.28}$$

where the sum is over a finite set of k and b. Then

$$f(x) = -\sum_{k,b} (-1)^k \frac{c_{k,b}}{k!} x^{-b} \log^k x + O(x^{-M}) \tag{9.29}$$

as $x \to \infty$.

(ii) [ASYMPTOTICS FOR $x \to 0$] *Assume that $f^*(s)$ admits a meromorphic continuation to $\Re(s) \in (-M, \beta)$ for some $-M < \alpha$ such that (9.27) and (9.28) hold in $\Re(s) \in (-M, \beta)$. Then*

$$f(x) = \sum_{k,b} (-1)^k \frac{c_{k,b}}{k!} x^{-b} \log^k x + O(x^M) \tag{9.30}$$

as $x \to 0$.

Proof. Part (i) we have already discussed above. It suffices to follow these steps tailored to (9.28). For part (ii) it is enough to observe that by (9.6) $\mathcal{M}[f(1/x); s] = -\mathcal{M}[f(x); s]$. An alternative proof is possible by following the above steps and expanding the Mellin integral to the left instead to the right as we did for $x \to \infty$. ■

Example 9.3 *A Recurrence Arising the Analysis of the PATRICIA Height*

In [215] Knessl and Szpankowski analyzed the height of PATRICIA tries. Among others, the following equation arose

$$C_n = \frac{2^{2-n}}{n+1} \sum_{k=2}^{n} \binom{n+1}{k} C_k, \qquad n \geq 2 \tag{9.31}$$

with $C_2 = 1$. Certainly, the exponential generating function $C(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} C_n$ of C_n satisfies

$$C(z) = 8C\left(\frac{z}{2}\right) \frac{e^{z/2} - 1}{z}.$$

Setting $\widetilde{C}(z) = e^{-z} C(z)$ (so that $\widetilde{C}(z)$ is the Poisson transform of C_n) we arrive at

$$\widetilde{C}(z) = \frac{8}{z}(1 - e^{-z/2}) \widetilde{C}\left(\frac{z}{2}\right).$$

Observe that $\widetilde{C}(z) \sim z^2/2$ as $z \to 0$ since $C_2 = 1$ and we define $C_0 = C_1 = 0$. Next we set $\widetilde{C}(z) = z^2 G(z)/2$ and replace z by $2z$ in the above, which gives

$$G(2z) = G(z) \left(\frac{1 - e^{-z}}{z}\right)$$

with $G(0) = 1$, and after another change $F(z) = \log G(z)$ we obtain

$$F(2z) - F(z) = \log \left(\frac{1 - e^{-z}}{z}\right),$$

with $F(0) = 0$. The above functional equation is manageable by Mellin transforms. The Mellin transform $F^*(s)$ of $F(z)$ is

$$F^*(s) = \frac{2^s}{1 - 2^s} \mathcal{M}\left[\log \left(\frac{1 - e^{-z}}{z}\right)\right]$$

for $\Re(s) \in (-1,0)$. In Exercise 1 we asked the reader to prove Entry 13 of Table 9.1, that is,

$$\mathcal{M}\left[\frac{z}{e^z-1}-1\right] = \Gamma(s+1)\zeta(s+1), \qquad -1 < \Re(s) < 0$$

leading to

$$F^*(s) = -\frac{\Gamma(s+1)\zeta(s+1)}{s(2^{-s}-1)}$$

for $\Re(s) \in (-1,0)$. Taking into account the singularities of $F^*(s)$, that is, a triple pole at $s=0$ and simple poles along the imaginary axis at $\chi_k = -2\pi i k/\log 2$, and using the reverse mapping theorem, we arrive at

$$\exp[F(z)] \sim \sqrt{z}2^{-1/12}\exp\left(\frac{\gamma(1)+\gamma^2/2-\pi^2/12}{\log 2}\right)\exp\left[-\frac{1}{2}\frac{\log^2(z)}{\log 2}+\Psi(\log_2 z)\right] \tag{9.32}$$

where

$$\Psi(\log_2 z) = \sum_{\substack{\ell=-\infty \\ \ell\neq 0}}^{\infty} \frac{1}{2\pi i\ell}\Gamma\left(1-\frac{2\pi i\ell}{\log 2}\right)\zeta\left(1-\frac{2\pi i\ell}{\log 2}\right)e^{2\pi i\ell\log_2 z}.$$

The above is true for $z\to\infty$ along the real axis, but we can analytically continue it to a cone around the real positive axis. In Section 9.3 below we extend the Mellin transform to the complex plane, thus obtaining automatically the same conclusion. However, to obtain asymptotics of C_n one needs another approach. The singularity analysis discussed in Section 8.3.2 does not work well in this case. In the next chapter we develop a new tool called the analytic depoissonization that will handle it easily. □

Not surprisingly, there exists a direct mapping from asymptotics of $f(x)$ near zero and infinity to a singular expansion of $f^*(s)$. We formulate it next.

Theorem 9.3 (Direct Mapping) *Let f be continuous with Mellin transform $f^*(s)$ defined in a non-empty strip $\langle\alpha,\beta\rangle$.*

(i) *Assume that $f(x)$ admits near $x\to 0$ the following asymptotic expansion*

$$f(x) = \sum_{k,b} c_{k,b}x^b\log^k x + O(x^M) \tag{9.33}$$

where $-M < -b \le \alpha$. Then $f^*(s)$ is continuable to the strip $\langle -M,\beta\rangle$ and admits the singular expansion

$$f^*(s) \asymp \sum_{k,b} c_{k,b}\frac{(-1)^k k!}{(s+b)^{k+1}} \tag{9.34}$$

for $-M < \Re(s) < \beta$.

(ii) *Let $f(x)$ have the following asymptotic expansion for $x \to \infty$*

$$f(x) = \sum_{k,b} c_{k,b} x^{-b} \log^k x + O(x^{-M}) \tag{9.35}$$

where $\beta \le b < M$. Now, the Mellin transform $f^*(s)$ can be continued to the strip $\langle \alpha, M \rangle$ and

$$f^*(s) \asymp -\sum_{k,b} c_{k,w} \frac{(-1)^k k!}{(s-b)^{k+1}}. \tag{9.36}$$

for $\alpha < \Re(s) < M$.

Proof. As before we need only to concentrate on one case. For a change, we prove the case $x \to 0$. We shall follow Flajolet *et al.* [115]. By assumption the function

$$g(x) = f(x) - \sum_{k,b} c_{k,b} x^b \log^k x$$

is $O(x^M)$. In the fundamental strip we also have

$$f^*(s) = \int_0^1 g(x) x^{s-1} + \int_0^1 \sum_{k,b} c_{k,b} x^{s+b-1} \log^k x + \int_1^\infty f(x) x^{s-1} dx.$$

The first integral above is defined in $\langle -M, \infty \rangle$ since $g(x) = O(x^M)$ for $x \to 0$. The third integral is analytic for $\Re(s) < \beta$, while the second integral is easily computable (cf. Table 9.1). In summary, $f^*(s)$ exists in $\langle -M, \beta \rangle$ and has (9.34) as its singular expansion. ∎

The direct and reverse mapping theorems are at the heart of the Mellin transform method. They have a myriad of applications. The survey by Flajolet, Gourdon and Dumas [115] is the best source of information. Also Chapter 7 of the forthcoming book by Flajolet and Sedgewick has many interesting examples. Here, we just illustrate it on an example and a few exercises. We shall return to it in the next chapter.

Example 9.4 *Generalized Euler-Maclaurin Summation Formula*

In Section 8.2.1 we discussed the Euler-Maclaurin summation formula that allows us to approximate discrete sums by integrals. Let us now consider

$$F(x) = \sum_{n=1}^{\infty} f(nx).$$

where

$$f(x) = \sum_{k=0}^{\infty} c_k x^k, \qquad x \to 0$$

and for simplicity of presentation we postulate $f(x) = O(x^{-3/2})$ as $x \to \infty$. The Mellin transform $F^*(s)$ of $F(x)$ is

$$F^*(s) = f^*(s)\zeta(s)$$

for $\Re(s) < 1$. Taking into account the pole of $\zeta(s)$ at $s = 1$, we can write $F^*(s)$ as the following singular expansion

$$F^*(s) \asymp \frac{f^*(1)}{s-1} + \sum_{k=0}^{\infty} \frac{c_k \zeta(-k)}{s+k},$$

due to our assumptions regarding $f(x)$ as $x \to 0$. By the reverse mapping theorem we find

$$\begin{aligned}\sum_{n=1}^{\infty} f(nx) &\sim \frac{1}{x}\int_0^{\infty} f(t)dt + \sum_{k=1}^{\infty} c_k \zeta(-k) x^k, \\ &= \frac{1}{x}\int_0^{\infty} f(t)dt - \frac{1}{2}f(0) - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k} c_{2k-1} x^{2k-1}\end{aligned}$$

where B_k are Bernoulli numbers (cf. Table 8.1). The last last line above is a consequence of $\zeta(-2k+1) = -B_{2k}/2k$.

The above is basically the Euler-Maclaurin formula. Interestingly, it can be generalized to fractional exponents of x. For example, let us consider

$$F(x) = \sum_{n=1}^{\infty} \log n f(nx)$$

where

$$f(x) = \sum_{k=0}^{\infty} c_k x^{\beta_k}, \qquad x \to 0$$

with $-1 < \beta_0 < \beta_1 < \cdots$. This translates to

$$F^*(s) = -f^*(s)\zeta'(s).$$

The singular expansion, due to the double pole of $\zeta'(s)$ at $s = 1$, is

$$F^*(s) \asymp \frac{f^*(1)}{(s-1)^2} + \frac{f^{*'}(1)}{s-1} - \sum_{k=0}^{\infty} \frac{c_k \zeta'(-\beta_k)}{s+\beta_k}.$$

By the reverse mapping theorem we finally arrive at

$$F(x) \sim \frac{1}{x}\log\frac{1}{x}\int_0^{\infty} f(t)dt + \frac{1}{x}\int_0^{\infty} f(t)\log t dt - \sum_{k=0} c_k \zeta'(-\beta_k) x^{\beta_k}. \tag{9.37}$$

Using this, we can easily derive the Ramanujan formula, namely,

$$\sum_{k=1}^{\infty} e^{-kx} \log k = \frac{1}{x}\left(\log\frac{1}{x} - \gamma\right) + \log\sqrt{2\pi} + O(x).$$

In Exercises 6-9 we propose some more examples and generalizations. □

9.3 Extension to the Complex Plane

In various applications of the Mellin transform one needs an extension of the transform to complex analytic functions. To motivate this, let us consider a generic example. Let f_n be a sequence such that it has the generating function $f(z)$ satisfying, say, the following functional equation

$$f(z) = a(z) + f(pz) + f(qz) \tag{9.38}$$

for some $a(z)$ and $p + q = 1$. If z is real, the Mellin transform $f^*(s)$ becomes

$$f^*(s) = \frac{a^*(s)}{1 - p^{-s} - q^{-s}}. \tag{9.39}$$

where $a^*(s)$ is the Mellin transform of $a(z)$. This can be used to obtain asymptotics of $f(x)$ for $x \to 0$ or $x \to \infty$. Assume that we obtained $f(x) = 1/\sqrt{1-x} + O(1)$, and would like to infer asymptotics of f_n through singularity analysis. For this, however, we need *an extension* of $f(x)$ to complex z which further would require an extension for the Mellin transform to complex analytic functions.

We discuss such an extension in this section. Its application will be illustrated in Section 9.4.2 and the next chapter. We extend the Mellin transform to complex functions $F(z)$ such that $f(x) = F(z)\mid_{z=x>0}$ for z complex.

Theorem 9.4 *Let (θ_1, θ_2) be an interval containing 0. Let $F(z)$ be an analytic function defined for z in a cone $S_{\theta_1,\theta_2} = \{z : \theta_1 < \arg(z) < \theta_2\}$ such that $F(z) = O(z^{-c})$ for $z \to 0$, and $F(z) = O(z^{-d})$ for $z \to \infty$ with z confined to the cone (or more generally: there is an interval (c, d) such that for all $b \in (c, d)$: $F(z) = O(z^{-b})$ as $z \to 0$ and $z \to \infty$ in the cone).*

(i) *Let $\mathcal{L}$ be a curve that starts at $z = 0$ and goes to ∞ inside the cone $\mathcal{S}_{\theta_1,\theta_2}$. For $\Re(s) \in (c, d)$ the integral*

$$\mathcal{M}(F, s) = F^*(s) := \int_{\mathcal{L}} F(z) z^{s-1} dz$$

exists and does not depend on the curve $\mathcal{L}$. We call it the **complex Mellin transform**.

(ii) *The complex Mellin transform $F^*(s)$ is identical to the real Mellin transform $f^*(s)$, that is,*

$$\int_0^\infty F(x) x^{s-1} dx = f^*(s) = F^*(s) = \int_{\mathcal{L}} F(z) z^{s-1} dz \ .$$

Furthermore, $\mathcal{M}(F(za), s) = a^{-s} f^*(s)$ defined on $S_{\theta_1 - \arg(a), \theta_2 - \arg(a)}$ with $a \in \mathcal{S}_{\theta_1,\theta_2}$.

(iii) *For $b \in (c, d)$ and $\theta_1 < \theta < \theta_2$, $f^*(b + it)e^{\theta t}$ is absolutely integrable with respect to the variable t from $-\infty$ to $+\infty$. The inverse Mellin exists and becomes*

$$F(z) = \frac{1}{2i\pi} \int_{\Re(s)=b} z^{-s} f^*(s) ds$$

for $b \in (c, d)$.

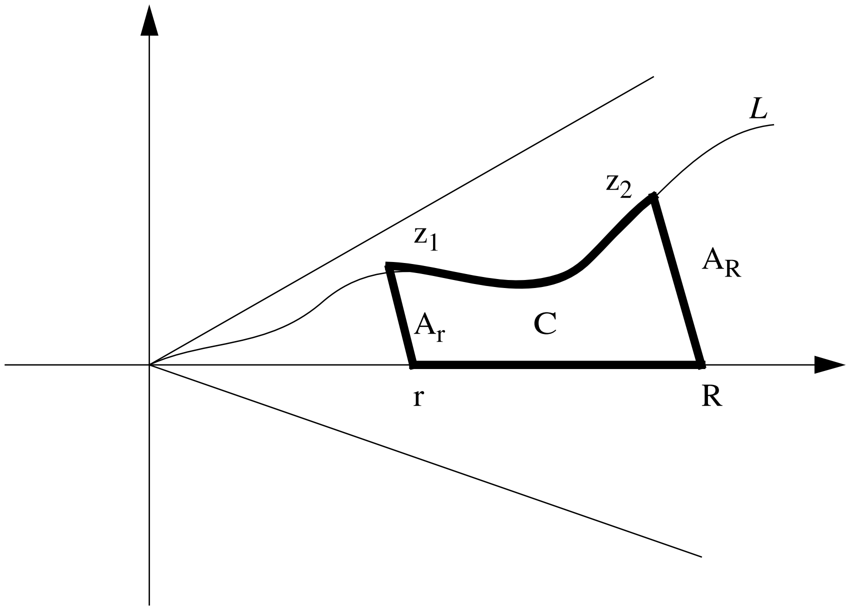

Figure 9.2: Contours Used in the Proof of Theorem 9.4.

Proof: For part (i), observe that the function $F(z)z^{s-1}$ is analytic, thus by Cauchy's formula

$$\oint F(z)z^{s-1}dz = 0$$

where the integration is over any arbitrary closed curve inside the cone $\mathcal{S}_{\theta_1,\theta_2}$. Let z_1 and z_2 be two points on $\mathcal{L}_0^\infty$, let $r = |z_1|$ and $R = |z_2|$ and let $\mathcal{L}_r^R$ be the part of $\mathcal{L}_0^\infty$ between z_1 and z_2. Furthermore, we denote by A_r and A_R the circular arcs connecting z_1 with r and z_2 with R. We integrate along the contour $\mathcal{C}$ formed by $\mathcal{L}_r^R$, the subinterval (r, R) of the real line and the two arcs A_r and A_R (cf. Figure 9.2). We have

$$\int_{\mathcal{L}_r^R} F(z)z^{s-1}dz = \int_r^R F(z)z^{s-1}dz + \int_{A_R} F(z)z^{s-1}dz + \int_{A_r} F(z)z^{s-1}dz \tag{9.40}$$

For $R \to \infty$ and $r \to 0$ we observe that $\int_{A_R} F(z)z^{s-1}dz = O(R^{\Re(s)-b})$ for any $b \in (\Re(s), d)$, and $\int_{A_r} F(z)z^{s-1}dz = O(r^{\Re(s)-a})$ for $a \in (c, \Re(s))$, respectively. Thus passing to the limits as $R \to \infty$ and $r \to 0$, we conclude that the contribution of the arcs vanish. This proves part (i) and the first part of (ii).

For part (ii), we infer immediately from (9.40), that is,

$$F^*(s) = \lim_{R\to\infty} \int_{\mathcal{L}_0^R} F(z)z^{s-1}dz = \lim_{R\to\infty} \int_0^R F(z)z^{s-1}dz = f^*(s) \ .$$

To prove the identity $\mathcal{M}(F(za), s) = a^{-s} f^*(s)$ one only needs to make the change of variable $za = z'$ in the integration. Notice that the function $F(za)$ is now defined in the cone $S_{\theta_1 - \arg(a), \theta_2 - \arg(a)}$.

Point (iii) is a natural consequence of point (ii) by selecting $a = e^{i\theta}$. In this case, the (real) Mellin transform of the function $F(xe^{-i\theta})$ becomes $e^{s\theta} f^*(s)$ which is absolutely integrable. Therefore, for all $z \in \mathcal{S}_{\theta_1, \theta_2}$ the function $z^{-s} f^*(s)$ is absolutely integrable with respect to $\Im(s)$. Finally, we conclude that the Mellin inverse identity holds in the complex case. ∎

The above extension enjoys all the "nice" properties of Mellin transforms of real x that we discussed so far. In particular, it can be used to obtain an asymptotic expansion of $F(z)$ for $z \to 0$ and $z \to \infty$.

Theorem 9.5 *Let $f(x)$ be a function of real variable. Let $f^*(s)$ be its Mellin transform defined in the strip $\Re(s) \in (c, d)$. Let (θ_1, θ_2) be an interval containing* 0. *If for all $b \in (c, d)$ function $f^*(b + it)e^{\theta_1 t}$ and $f^*(b + it)e^{\theta_2 t}$ are absolutely integrable, then*
(i) *The analytic continuation $F(z)$ of $f(x)$ exists in cone $\mathcal{S}_{\theta_1, \theta_2}$ and $F(z) = O(z^{-b})$ for $|z| \to 0$ and $|z| \to \infty$ for all $b \in (c, d)$.*

(ii) *If the Mellin transform $f^*(s)$ has singularities that are regular isolated poles in the strip $\Re(s) \in (c', d')$ where $c' < c$ and $d' > d$, then the function $F(z)$ has the same asymptotic expansions as $f(x)$ up to $z^{-c'}$ at 0 and up to order $z^{-d'}$ at ∞.*

Proof: Defining

$$F(z) = \frac{1}{2\pi i} \int_{\Re(s)=b} z^{-s} f^*(s) ds$$

one introduces an analytic function that matches $f(x)$ on the real line. By an elementary estimate of the integrand we have $F(z) = O(z^{-b})$ due to the assumption concerning the smallness of $f^*(b+it)$ as $t \to \pm\infty$. Part (ii) is a simple consequence of (i), the fact that $f^*(s)$ is the complex Mellin transform of $F(z)$, and the inverse mapping theorem. ∎

To illustrate the above let us return to the functional equation (9.38) that has solution (9.38). One can obtain asymptotics of $f(z)$ by the residue theorem. In particular if $a^*(s) = \alpha(s)\Gamma(s)$ with $\alpha(s)$ of at most polynomial growth for $\pm i\infty$, then the asymptotic expansion remain valid in all cone strictly included in $\mathcal{S}_{-\pi/2, \pi/2}$ because $|\Gamma(it)|^2 = \pi(t \sinh(\pi t))^{-1} = O(e^{-\pi|t|})$ when real t tends to $\pm\infty$.

9.4 Applications

We are again in the applications section. We shall derive here the variance of the depth in a generalized digital search tree, and the asymptotic expansion of the minimax redundancy for renewal processes. This last application is probably one of the most complicated examples discussed in this book; to solve this problem we will need all the knowledge we learned so far.

9.4.1 Variance of the Depth in a Generalized Digital Search Tree

As in Section 8.7.3 we study here the depth in a digital search tree. However, this time we consider a generalized digital search tree in which every node can store up to b strings, known also as b-DST. The depth is denoted as D_m and we compute here the variance of D_m for the unbiased binary memoryless source in which m independent strings are generated each being a sequence of zeros and ones occurring with the same probability. As before, let $\bar{B}_m(k)$ be the *expected* number of strings on level k of a b-DST tree. We know that $\Pr\{D_m = k\} = \bar{B}_m(k)/m$ (cf. (8.159). Let $B_m(u) = \sum_{k\geq 0} \bar{B}_m(k)u^k$ be the generating function of $\bar{B}_m(k)$. As in Section 8.7.3, we observe that the Poisson generating function $\widetilde{B}(u,z) = e^{-z}\sum_{m=0}^{\infty} B_m(u)\frac{z^m}{m!}$ satisfies

$$\left(1+\frac{\partial}{\partial z}\right)^b \widetilde{B}(u,z) = b + 2u\widetilde{B}(u,z/2) \tag{9.41}$$

where $\left(1+\frac{\partial}{\partial z}\right)^b := \sum_{i=0}^{b}\binom{b}{i}\frac{\partial^i}{\partial z^i}$. The coefficients of $\widetilde{B}(u,z)$ can be computed by solving a linear recurrence discussed in Section 7.6.1 and Section 8.7.3. Unfortunately, there is no easy way to solve such a recurrence unless $b = 1$ (cf. Section 7.6.1). To circumvent this difficulty, Flajolet and Richmond [113] reduced it to a certain functional equation on an *ordinary* generating function that is easier to solve. We proceed along this path.

We define a sequence $g_k(u) = k![z^k]\widetilde{B}(u,z)$, that is, $\widetilde{B}(u,z) = \sum_{k=0}^{\infty} g_k(u)\frac{z^k}{k!}$. Observe that $B_m(u) = \sum_{k=0}^{m}\binom{m}{k}g_k(u)$. Let $G(u,z) = \sum_{k=0}^{\infty} g_k(u)z^k$. We also need the ordinary generating function of $B_k(u)$ denoted as

$$F(u,z) = \sum_{k=0}^{\infty} B_k(u)z^k.$$

As in Example 11 of Chapter 7 we have

$$F(u,z) = \frac{1}{1-z}G\left(u,\frac{z}{1-z}\right) .$$

that implies

$$F_u^{(k)}(u,z) = \frac{1}{1-z}G_u^{(k)}\left(u,\frac{z}{1-z}\right) , \tag{9.42}$$

where $F_u^{(k)}(z,u)$ denotes the kth derivative of $F(z,u)$ with respect to u. Then from (9.41) we obtain

$$G(u,z)(1+z)^b = z(1+z)^b - z^{b+1} + 2uz^bG(u,\frac{z}{2}) , \tag{9.43}$$

$$G_u'(u,z)(1+z)^b = 2z^bG(u,\frac{z}{2}) + 2uz^bG_u'(u,\frac{z}{2}) , \tag{9.44}$$

$$G_u''(u,z)(1+z)^b = 4z^bG_u'(u,\frac{z}{2}) + 2uz^bG_u''(u,\frac{z}{2}) . \tag{9.45}$$

In order to compute the variance, we need $L^{\underline{1}}(z) := G'_u(u,z)|_{u=1}$ and $L^{\underline{2}}(z) := G''_u(u,z)|_{u=1}$. From (9.44) and (9.45) we immediately obtain

$$\begin{aligned} L^{\underline{1}}(z)(1+z)^b &= z^{b+1} + 2z^b L^{\underline{1}}(\frac{z}{2}), \\ L^{\underline{2}}(z)(1+z)^b &= 4z^b L^{\underline{1}}(\frac{z}{2}) + 2z^b L^{\underline{2}}(\frac{z}{2}). \end{aligned}$$

Iterating these equations we easily find (cf. [113] and Section 7.6.1)

$$L^{\underline{1}}(z) = \sum_{k=0}^{\infty} \frac{(2z^b)(2(\frac{z}{2})^b)\cdots(2(\frac{z}{2^k})^b)}{\left((1+z)(1+\frac{z}{2})\cdots(1+\frac{z}{2^k})\right)^b} \frac{z}{2^{k+1}}, \tag{9.46}$$

$$L^{\underline{2}}(z) = \sum_{k=0}^{\infty} \frac{(2z^b)(2(\frac{z}{2})^b)\cdots(2(\frac{z}{2^k})^b)}{\left((1+z)(1+\frac{z}{2})\cdots(1+\frac{z}{2^k})\right)^b} 2L^{\underline{1}}(\frac{z}{2^{k+1}}). \tag{9.47}$$

The next step is to transform the above sums (9.46)–(9.47) into harmonic sums. For this, we set $z = 1/t$ and define $Q(t) = \prod_{k=0}^{\infty}(1+\frac{t}{2^k})$. Then (9.46)–(9.47) become

$$\frac{tL^{\underline{1}}(\frac{1}{t})}{\left(Q(\frac{t}{2})\right)^b} = \sum_{k=0}^{\infty} \frac{1}{\left(Q(2^k t)\right)^b}, \tag{9.48}$$

$$\frac{tL^{\underline{2}}(\frac{1}{t})}{\left(Q(\frac{t}{2})\right)^b} = 2\sum_{k=0}^{\infty} \frac{2^{k+1} t L^{\underline{1}}(\frac{1}{2^{k+1}t})}{\left(Q(\frac{2^{k+1}t}{2})\right)^b}. \tag{9.49}$$

which are harmonic sums. The Mellin transforms are

$$\begin{aligned} \mathcal{M}\left[\frac{tL^{\underline{1}}(\frac{1}{t})}{Q^b(\frac{t}{2})}; s\right] &= \frac{1}{1-2^{-s}} I(s), \\ \mathcal{M}\left[\frac{tL^{\underline{2}}(\frac{1}{t})}{Q^b(\frac{t}{2})}; s\right] &= \frac{2^{1-s}}{(1-2^{-s})^2} I(s), \end{aligned}$$

where

$$I(s) = \int_0^{\infty} \frac{t^{s-1}}{Q^b(t)} dt = \frac{\pi}{\sin \pi s} J(s), \tag{9.50}$$

$$J(s) = \frac{1}{2\pi i} \int_{\mathcal{H}} \frac{(-t)^{s-1}}{Q^b(t)} dt \tag{9.51}$$

with $\mathcal{H}$ being the Hankel contour.

Applying the reverse mapping Theorem 9.2 we find that

$$\begin{aligned} L^{\underline{1}}(\frac{1}{t}) &= \frac{1}{t}k(t) + bk(t) + O(t\log t^{-1}), \\ L^{\underline{2}}(\frac{1}{t}) &= \frac{1}{t}K(t) + bK(t) + O(t\log^2 t^{-1}) \ , \end{aligned}$$

where

$$\begin{aligned} k(t) &= \log_2\frac{1}{t} + \frac{1}{2} + \frac{J'(0)}{\log 2} - \frac{1}{\log 2}\sum_{k=0}\frac{I(\chi_k)}{\chi_k}t^{-\chi_k}, \\ K(t) &= \log_2^2\frac{1}{t} + \frac{2J'(0)}{\log^2 2}\log\frac{1}{t} - \left(\frac{1}{6} + \frac{J''(0)}{\log^2 2} - \frac{\pi^2}{3\log^2 2}\right) + 8bt \\ &- \frac{2}{\log^2 2}\sum_{k=0}\frac{I(\chi_k)}{\chi_k}t^{-\chi_k}\log\frac{1}{t} + \frac{2}{\log^2 2}\sum_{k=0}\left(\frac{I(\chi_k)}{\chi_k^2} - \frac{I'(\chi_k)}{\chi_k}\right)t^{-\chi_k} \ , \end{aligned}$$

with $\chi_k = 2\pi ik/\log 2$ for $k = 0, \pm 1, \ldots$ being the roots of $1 - 2^{-s} = 0$. The next step is to recover asymptotic expansions of $F_u'(1,z)$ and $F_u''(1,z)$ from which computing the mean $\mathbf{E}[D_m]$ and the variance $\mathbf{Var}[D_m]$ is routine. Using (9.42) and changing $t = 1/z$, the above translates to

$$\begin{aligned} F_u'(1,z) &= \frac{z}{(1-z)^2}\log_2\frac{z}{1-z} + \frac{1}{\log 2}\left(\frac{\log 2}{2} + J'(0)\right)\frac{z}{(1-z)^2} \\ &- \frac{1}{\log 2}\sum_{k\neq 0}\frac{I(\chi_k)}{\chi_k}\frac{z^{1+\chi_k}}{(1-z)^{2+\chi_k}} + \frac{b}{\log 2}\frac{1}{1-z}\left(\log\frac{z}{1-z}\right. \\ &+ \left.\frac{\log 2}{2} + J'(0) - \sum_{k\neq 0}\frac{I(\chi_k)}{\chi_k}\frac{z^{\chi_k}}{(1-z)^{\chi_k}}\right) + O\left(\log\frac{1}{(1-z)}\right) \\ F_{uu}''(1,z) &= \frac{z}{(1-z)^2}\log_2^2\frac{z}{1-z} + \frac{2J'(0)}{\log^2 2}\frac{z}{(1-z)^2}\log\frac{z}{1-z} \\ &- \left(\frac{1}{6} + \frac{J''(0)}{\log^2 2} - \frac{\pi^2}{3\log^2 2}\right)\frac{z}{(1-z)^2} - \frac{2}{\log^2 2}\sum_{k\neq 0}\frac{I(\chi_k)}{\chi_k}\frac{z^{\chi_k}}{(1-z)^{\chi_k}}\log\frac{z}{1-z} \\ &+ \frac{2}{\log^2 2}\sum_{k\neq 0}\left(\frac{I(\chi_k)}{\chi_k^2} - \frac{I'(\chi_k)}{\chi_k}\right)\frac{z^{\chi_k}}{(1-z)^{\chi_k}} + O\left(\frac{\log^2(1-z)}{(1-z)}\right) \end{aligned}$$

Having this, we apply the singularity analysis to recover the mean and the variance (through the second factorial moment $\mathbf{E}[D_m(D_m-1)]$). However, strictly speaking we derived the above *only* for z real (since the Mellin transform is defined for z real) while we need an extension to the complex plane in order to apply Theorem 8.13 of the singularity analysis. Luckily enough, the extension of Mellin transforms to the complex plane was already presented in Section 9.3. With this in mind, we are allowed to apply the singularity analysis, in particular Theorem 8.12, and after some algebra, we find that

$$\begin{aligned}
\mathbf{E}[D_m] &= \log_2 m + \frac{1}{\log 2}\left((\gamma-1)+\frac{\log 2}{2}+J'(0)-\sum_{k\neq 0}\frac{I(\chi_k)}{\chi_k}\frac{m^{\chi_k}}{\Gamma(2+\chi_k)}\right)\\
&+ O\left(\frac{\log m}{m}\right)\\
\mathbf{Var}[D_m] &= \frac{1}{12}+\frac{1}{\log^2 2}\left(1+\frac{\pi^2}{6}\right)+\frac{1}{\log^2 2}\left(J''(0)-(J'(0))^2\right)+\frac{1}{\log 2}\delta_2(\log_2 m)-[\delta_1^2]_0\\
&+ O\left(\frac{\log^2 m}{m}\right)
\end{aligned}$$

where

$$J'(0) = \int_0^1\left(\frac{1}{Q(t)^b}-1\right)\frac{dt}{t}+\int_0^\infty\frac{1}{Q(t)^b}\frac{dt}{t}, \tag{9.52}$$

$$J''(0) = -\frac{\pi^2}{3}+2\int_0^1\left(\frac{1}{Q(t)^b}-1\right)\frac{\log t}{t}dt+2\int_0^\infty\frac{1}{Q(t)^b}\frac{\log t}{t}dt, \tag{9.53}$$

$\delta_2(\cdot)$ is a periodic function with mean zero and period 1, and $[\delta_1^2]_0$ is a very small constant (e.g., $[\delta_1^2]_0 \le 10^{-10}$ for $b=1$). More precisely, (cf. Hubalek [169])

$$[\delta_1^2]_0 = \frac{1}{L^2}\sum_{k\neq 0}\frac{I(\chi_k)I(-\chi_k)}{\Gamma(2+\chi_k)\Gamma(2-\chi_k)},$$

and

$$I(\chi_k) = \frac{1}{\chi_k}+\int_0^1\left(Q^{-b}(t)-1\right)t^{\chi_k-1}dt+\int_1^\infty Q^{-b}(t)t^{\chi_k}dt.$$

Numerical evaluation of $J'(0)$ and $J''(0)$ can be found in Hubalek [169].

9.4.2 Redundancy of Renewal Processes

We evaluate here the minimax redundancy of a renewal process. We shall follow Flajolet and Szpankowski [123]. The reader is referred to Section 8.7.2 for a brief review of the redundancy problem. We start with a precise definition of the class $\mathcal{R}_0$ of renewal processes and its associated sources. Let $T_1, T_2, \ldots$ be a sequence of i.i.d. (i.e., independently, identically distributed) nonnegative–valued random variables with distribution $Q(j) = \Pr\{T_1 = j\}$. An independent random variable T_0 is introduced with distribution

$$\Pr\{T_0 = i\} = \mathbf{E}[T_1]^{-1}\sum_{j\ge i}Q(j) \tag{9.54}$$

provided $\mathbf{E}[T_1] < \infty$. The quantities $\{T_i\}_{i=1}^{\infty}$ are the interarrival times, while T_0 is the initial waiting time. The process $T_0, T_0+T_1, T_0+T_1+T_2, \ldots$ is then called a renewal process and it is stationary whenever T_0 has the distribution above with $\mathbf{E}[T_1] < \infty$. With such a renewal process there is associated a *binary renewal sequence* that is a $0,1$-sequence in which the 1's occur exactly at the renewal epochs T_0, T_0+T_1, $T_0+T_1+T_2$, etc.

By Shtarkov's method discussed in Section 8.7.2 we conclude that the minimax redundancy $R_n^*(\mathcal{R}_0)$ is well approximated by

$$R_n' := \log_2 \left(\sum_{x_1^n} \sup_Q P(x_1^n) \right).$$

In fact, $R_n' \le R_n^*(\mathcal{R}_0) \le R_n' + 1$. Instead of working with R_n' we evaluate $R_n = R_n' + O(1)$ or equivalently

$$r_n = 2^{R_n}.$$

It turns out that the problem of estimating r_n, reduces to the evaluation of the following combinatorial sum. A proof of the lemma below can be found in [123].

Lemma 9.6 *The quantity $r_n = 2^{R_n}$ admits the combinatorial form*

$$\begin{cases} r_n &= \displaystyle\sum_{k=0}^{n} r_{n,k} \\ r_{n,k} &= \displaystyle\sum_{\mathcal{P}(n,k)} \binom{k}{k_0 \cdots k_{n-1}} \left(\frac{k_0}{k}\right)^{k_0} \cdots \left(\frac{k_{n-1}}{k}\right)^{k_{n-1}} \end{cases}$$

where $\mathcal{P}(n,k)$ denotes the set of partitions of n into k summands, that is, the collection of tuples of nonnegative integers $(k_0, k_1, \ldots, k_{n-1})$ satisfying

$$n = k_0 + 2k_1 + \cdots + nk_{n-1}, \tag{9.55}$$

$$k = k_0 + k_1 + \cdots + k_{n-1} \tag{9.56}$$

for all $n \ge 0$ and $k \le n$.

Our goal is to estimate asymptotically r_n through asymptotics of $r_{n,k}$. A difficulty of finding such asymptotics stems from the factor $k!/k^k$ present in the definition of $r_{n,k}$. We circumvent this problem by analyzing a related pair of sequences, namely s_n and $s_{n,k}$ that are defined as

$$\begin{cases} s_n &= \displaystyle\sum_{k=0}^{n} s_{n,k} \\ s_{n,k} &= e^{-k} \displaystyle\sum_{\mathcal{P}(n,k)} \frac{k_0^{k_0}}{k_0!} \cdots \frac{k_{n-1}^{k_{n-1}}}{k_{n-1}!}. \end{cases} \tag{9.57}$$

The translation from s_n to r_n is most conveniently expressed in probabilistic terms. Introduce the random variable K_n whose probability distribution is $s_{n,k}/s_n$, that is,

$$\varpi_n : \qquad \Pr\{K_n = k\} = \frac{s_{n,k}}{s_n}, \tag{9.58}$$

where ϖ_n denotes the distribution. Then Stirling's formula yields

$$\begin{aligned} \frac{r_n}{s_n} &= \sum_{k=0}^{n} \frac{r_{n,k}}{s_{n,k}} \frac{s_{n,k}}{s_n} = \mathbf{E}[(K_n)! K_n^{K_n} e^{-K_n}] \\ &= \mathbf{E}[\sqrt{2\pi K_n}] + O(\mathbf{E}[K_n^{-\frac{1}{2}}]). \end{aligned} \tag{9.59}$$

Thus, the problem of finding r_n reduces to asymptotic evaluations of s_n, $\mathbf{E}[\sqrt{K_n}]$ and $\mathbf{E}[K_n^{-\frac{1}{2}}]$. The heart of the matter is the following lemma which provides the necessary estimates. The somewhat delicate proof of Lemma 9.7 constitutes the core of the section and it is deferred till the end of it.

Lemma 9.7 *Let $\mu_n = \mathbf{E}[K_n]$ and $\sigma_n^2 = \mathbf{Var}[K_n]$, where K_n has the distribution ϖ_n defined above in (9.58). The following holds*

$$s_n \sim \exp\left(2\sqrt{cn} - \frac{7}{8}\log n + d + o(1)\right) \tag{9.60}$$

$$\mu_n = \frac{1}{4}\sqrt{\frac{n}{c}}\log\frac{n}{c} + o(\sqrt{n}) \tag{9.61}$$

$$\sigma_n^2 = O(n\log n) = o(\mu_n^2), \tag{9.62}$$

where $c = \pi^2/6 - 1$, $d = -\log 2 - \frac{3}{8}\log c - \frac{3}{4}\log \pi$.

Once the estimates of Lemma 9.7 are granted, the moments of order $\pm\frac{1}{2}$ of K_n follow by a standard argument based on concentration of the distribution ϖ_n as discussed in Chapter 5.

Lemma 9.8 *For large n*

$$\mathbf{E}[\sqrt{K_n}] = \mu_n^{1/2}(1 + o(1)) \tag{9.63}$$

$$\mathbf{E}[K_n^{-\frac{1}{2}}] = o(1). \tag{9.64}$$

where $\mu_n = \mathbf{E}[K_n]$.

Proof. We only prove (9.63) since (9.64) is obtained in a similar manner. The upper bound $\mathbf{E}[\sqrt{K_n}] \le \sqrt{\mathbf{E}[K_n]}$ follows by Jensen's inequality (2.8) and the concavity of the function

$\sqrt{x}$. The lower bound follows from concentration of the distribution. Chebyshev's inequality (2.6) and (9.62) of Lemma 9.7 entail, for any arbitrarily small $\varepsilon > 0$,

$$\Pr\{|K_n - \mu_n| > \varepsilon\mu_n\} \leq \frac{\mathbf{Var}[K_n]}{\varepsilon^2\mu_n^2} = \frac{\delta(n)}{\varepsilon^2}$$

where $\delta(n) \to 0$ as $n \to \infty$. Then

$$\begin{aligned}
\mathbf{E}[\sqrt{K_n}] &\geq \sum_{k \geq (1-\varepsilon)\mu_n} \sqrt{k}\Pr\{K_n \geq k\} \\
&\geq (1-\varepsilon)^{\frac{1}{2}}\mu_n^{1/2}\Pr\{K_n \geq (1-\varepsilon)\mu_n\} \\
&\geq (1-\varepsilon)^{\frac{1}{2}}\left(1 - \frac{\delta(n)}{\varepsilon^2}\right)\mu_n^{1/2}.
\end{aligned}$$

Hence for any $\eta > 0$ one has

$$\mathbf{E}[\sqrt{K_n}] > \mu_n^{1/2}(1-\eta)$$

provided n is large enough. This completes the proof. ∎

In summary, r_n and s_n are related by

$$\begin{aligned}
r_n &= s_n\mathbf{E}[\sqrt{2\pi K_n}](1+o(1)) \\
&= s_n\sqrt{2\pi\mu_n}(1+o(1)),
\end{aligned}$$

by virtue of (9.59) and Lemma 9.8. This leads to

$$\begin{aligned}
R_n &= \log_2 r_n = \log_2 s_n + \frac{1}{2}\log_2 \mu_n + \log_2\sqrt{2\pi} + o(1) && (9.65) \\
&= \frac{2}{\log 2}\sqrt{\left(\frac{\pi^2}{6} - 1\right)n} - \frac{5}{8}\log_2 n + \frac{1}{2}\log_2\log n + O(1). && (9.66)
\end{aligned}$$

To complete the proof of our main result (9.65) we need to prove **Lemma 9.7** which is discussed next. Let

$$\beta(z) = \sum_{k=0}^{\infty}\frac{k^k}{k!}e^{-k}z^k.$$

which by the Lagrange inversion formula is equal to

$$\beta(z) = \frac{1}{1 - T(ze^{-1})}$$

where $T(z)$ is the tree function discussed in Section 7.3.2 (e.g., see (7.30)). As a matter of fact, $\beta(z) = B(ze^{-1})$ where $B(z)$ is defined in (7.34).

The quantities s_n and $s_{n,k}$ of (9.57) have generating functions,

$$S_n(u) = \sum_{k=0}^{\infty} s_{n,k} u^k, \qquad S(z,u) = \sum_{n=0}^{\infty} S_n(u) z^n.$$

Then, since equation (9.67) involves convolutions of sequences of the form $k^k/k!$, we have

$$\begin{aligned} S(z,u) &= \sum_{\mathcal{P}(n,k)} z^{1k_0+2k_1+\cdots} \left(\frac{u}{e}\right)^{k_0+\cdots+k_{n-1}} \frac{k_0^{k_0}}{k_0!} \cdots \frac{k_{n-1}^{k_{n-1}}}{k_{n-1}!} \\ &= \prod_{i=1}^{\infty} \beta(z^i u) \end{aligned} \tag{9.67}$$

We also need access to the first moment $\mu_n = \mathbf{E}[K_n]$ and the second factorial moment $\mathbf{E}[K_n(K_n-1)]$. They are obtained as

$$\begin{aligned} s_n &= [z^n]S(z,1), \\ \mu_n &= \frac{[z^n]S'_u(z,1)}{[z^n]S(z,1)}, \\ \mathbf{E}[K_n(K_n-1)] &= \frac{[z^n]S''_{uu}(z,1)}{[z^n]S(z,1)} \end{aligned}$$

where $S'_u(z,1)$ and $S''_{uu}(z,1)$ represent the first and the second derivative of $S(z,u)$ at $u=1$.

We deal here with s_n that is accessible through its generating function,

$$S(z,1) = \prod_{i=1}^{\infty} \beta(z^i). \tag{9.68}$$

The behavior of the generating function $S(z,1)$ as $z \to 1$ is an essential ingredient of the analysis. We know that the singularity of the tree function $T(z)$ at $z = e^{-1}$ is of the square-root type as discussed in Example 10 of Chapter 8 (cf. also [58]). Hence, near $z=1$, $\beta(z)$ admits the singular expansion (cf. Example 10 of Chapter 8 or (8.158))

$$\beta(z) = \frac{1}{\sqrt{2(1-z)}} + \frac{1}{3} - \frac{\sqrt{2}}{24}\sqrt{1-z} + O(1-z).$$

We now turn to the infinite product asymptotics as $z \to 1^-$, with z real. Let $L(z) = \log S(z,1)$ and $z = e^{-t}$, so that

$$L(e^{-t}) = \sum_{k=1}^{\infty} \log \beta(e^{-kt}). \tag{9.69}$$

The Mellin transform technique discussed in this chapter provides an expansion for $L(e^{-t})$ around $t = 0$ (or equivalently $z = 1$) since the sum (9.69) is a *harmonic sum* paradigm discussed in this chapter. The Mellin transform $L^*(s) = \mathcal{M}[L(e^{-t}); s]$ of $L(e^{-t})$ computed by the harmonic sum formula for $\Re(s) \in (1, \infty)$ is

$$L^*(s) = \zeta(s)\Lambda(s)$$

where

$$\Lambda(s) = \int_0^\infty \log \beta(e^{-t}) t^{s-1} dt.$$

Now, by the direct mapping Theorem 9.3 the expansion of $\beta(z)$ at $z = 1$ implies

$$\log \beta(e^{-t}) = -\frac{1}{2} \log t - \frac{1}{2} \log 2 + O(\sqrt{t}),$$

so that, collecting local expansions,

$$\Lambda(s) \asymp \left[\Lambda(1) \frac{1}{s-1}\right]_{s=1} + \left[\frac{1}{2} \frac{1}{s^2} - \frac{1}{2} \frac{\log 2}{s}\right]_{s=0}.$$

On the other hand, classical expansion of the zeta function gives (cf. [2, 337])

$$\zeta(s) \asymp \left[\frac{1}{s-1} + \gamma\right]_{s=1} + \left[-\frac{1}{2} - s \log \sqrt{2\pi}\right]_{s=0}.$$

Term-wise multiplication then provides the singular expansion of $L^*(s)$:

$$L^*(s) \asymp \left[\frac{\Lambda(1)}{s-1}\right]_{s=1} + \left[-\frac{1}{4s^2} - \frac{\log \pi}{4s}\right]_{s=0}.$$

An application of the reverse mapping Theorem 9.2 allows us to come back to the original function,

$$L(e^{-t}) = \frac{\Lambda(1)}{t} + \frac{1}{4} \log t - \frac{1}{4} \log \pi + O(\sqrt{t}), \tag{9.70}$$

which translates into (after using $z = 1 - t + O(t^2)$)

$$L(z) = \frac{\Lambda(1)}{1-z} + \frac{1}{4} \log(1-z) - \frac{1}{4} \log \pi - \frac{1}{2}\Lambda(1) + O(\sqrt{1-z}). \tag{9.71}$$

This computation is finally completed by the evaluation of $c := \Lambda(1)$:

$$\begin{aligned} c = \Lambda(1) &= -\int_0^1 \log(1 - T(x/e)) \frac{dx}{x} \\ &= -\int_0^1 \log(1-t) \frac{(1-t)}{t} dt \qquad (x = te^{1-t}) \\ &= \frac{\pi^2}{6} - 1. \end{aligned}$$

In summary, we just proved that, as $z \to 1^-$,

$$S(z,1) = e^{L(z)} = a(1-z)^{\frac{1}{4}} \exp\left(\frac{c}{1-z}\right)(1+o(1)), \tag{9.72}$$

where $a = \exp(-\frac{1}{4}\log\pi - \frac{1}{2}c)$. So far, the main estimate (9.72) has been established as z tends to 1 from the left, by real values, but by Theorem 9.4 we know that it is true for *complex* t only constrained in such a way that $-\frac{\pi}{2} + \epsilon \le \arg(t) \le \frac{\pi}{2} - \epsilon$, for any $\epsilon > 0$. Thus, the expansion (9.72) actually holds true as $z \to 1$ in a sector, say, $|\arg(1-z)| < \frac{\pi}{4}$.

It remains to collect the information gathered on $S(z,1)$ and recover $s_n = [z^n]S(z,1)$ asymptotically. The inversion is provided by the Cauchy coefficient formula.

We shall use the following lemma.

Lemma 9.9 *For positive $A > 0$, and reals B and C, define $f(z) = f_{A,B,C}(z)$ as*

$$f(z) = \exp\left(\frac{A}{1-z} + B\log\frac{1}{1-z} + C\log\left(\frac{1}{z}\log\frac{1}{1-z}\right)\right). \tag{9.73}$$

Then, the nth Taylor coefficient of $f_{A,B,C}(z)$ satisfies asymptotically, for large n,

$$\begin{aligned}[z^n]f_{A,B,C}(z) &= \exp\left(2\sqrt{An} + \frac{1}{2}\left(B - \frac{3}{2}\right)\log n + C\log\log\sqrt{\frac{n}{A}}\right. \\ &\quad \left. -\frac{1}{2}\log\left(4\pi e^{-A}/\sqrt{A}\right)\right)(1+o(1)).\end{aligned} \tag{9.74}$$

Proof. Problems of this kind have been considered by Wright [356] and others who, in particular, justify in detail that the saddle point method works in similar contexts. Therefore, we only outline the proof here. The starting point (see Table 8.4 in Section 8.4) is Cauchy's formula

$$[z^n]f(z) = \frac{1}{2\pi i}\oint e^{h(z)}dz$$

where

$$h(z) = \log f_{A,B,C}(z) - (n+1)\log z.$$

In accordance with (SP1) of Table 8.4 one chooses a saddle point contour that is a circle of radius r defined by $h'(r) = 0$. Asymptotically, one finds

$$r = 1 - \sqrt{\frac{A}{n}} + \frac{B-A}{2n} + o(n^{-1}),$$

and

$$h(r) = 2A\sqrt{\frac{n}{A}} + B\log\left(\sqrt{\frac{n}{A}}\right) + C\log\log\left(\sqrt{\frac{n}{A}}\right) + \frac{1}{2}A + o(1).$$

The "range" $\delta = \delta(n)$ of the saddle point, where most of the contribution of the contour integral is concentrated asymptotically, is dictated by the order of growth of derivatives (cf. (SP2) of Table 8.4). Here, $h''(r) \approx n^{3/2}$, while $h'''(r) \approx n^2$, so that

$$\delta(n) = n^{-3/4}.$$

In accordance with requirement(SP3) of Table 8.4, tails are negligible since the function $\exp((1-z)^{-1})$ decays very fast when going away from the real axis. In the central region, the local approximation (SP4) applies, as seen by expansions near $z = 1$. Thus requirements (SP1), (SP2), (SP3), and (SP4) of Table 8.4 are satisfied, implying, by (SP5) of Table 8.4

$$[z^{n-1}]f(z) = \frac{1}{\sqrt{2\pi|h''(r)|}} e^{h(r)} \left(1 + o(1)\right).$$

Some simple algebra, using

$$h''(r) = 2n\sqrt{n/A}\,\left(1 + o(1)\right),$$

yields the stated estimate (9.74). ∎

Now, the function $S(z, 1)$ is only known to behave like $f(z)$ of Lemma 9.9 in the vicinity of 1. In order to adapt the proof of Lemma 9.9 and legitimate the use of the resulting formula, we need to prove that $S(z, 1)$ decays fast away from the real axis. This somewhat technical lemma below is proved in [123] and we omit here its derivation.

Lemma 9.10 (Concentration property) *Consider the ratio*

$$q(z) = \prod_{j=1}^{\infty} \left| \frac{\beta(z^j)}{\beta(|z|^j)} \right|.$$

Then, there exists a constant $c_0 > 0$ such that

$$q(re^{i\theta}) = O\left(e^{-c_0(1-r)^{-1}}\right),$$

uniformly, for $\frac{1}{2} \le r < 1$ and $|\arg(re^{i\theta} - 1)| > \frac{\pi}{4}$.

We are now eventually ready to return to the estimate of s_n in Lemma 9.7. In the region $|\arg(z-1)| < \frac{\pi}{4}$, the Mellin asymptotic estimates (9.70) and (9.72) apply. This shows that in this region,

$$S(z, 1) = e^{o(1)} f_{A,B,C}(z) \qquad (z \to 1),$$

where the function f is that of Lemma 9.9 and the constants A, B, C have the values assigned by (9.72):

$$A = c = \frac{\pi^2}{6} - 1, \quad B = -\frac{1}{4}, \quad C = 0.$$

In the complementary region, $|\arg(z-1)| > \frac{\pi}{4}$, the function $S(z,1)$ is exponentially smaller than $S(|z|,1)$ by Lemma 9.10. From these two facts, the saddle point estimates of Lemma 9.9 are seen to apply, by a trivial modification of the proof of that lemma. This concludes the proof of Equation (9.60) in Lemma 9.7.

It remains to complete the evaluation of μ_n and σ_n^2, following the same principles as before. Start with $\mu_n = \mathbf{E}[K_n]$, with the goal of establishing the evaluation (9.61) of Lemma 9.7. It is necessary to estimate $[z^n]S'_u(z,1)$, with

$$S'_u(z,1) = S(z,1)\sum_{k=0}^{\infty} z^k \frac{\beta'(z^k)}{\beta(z^k)}.$$

Let

$$D_1(z) = \sum_{k=0}^{\infty} \alpha(z^k), \qquad \text{where} \qquad \alpha(z) = z\frac{\beta'(z)}{\beta(z)}.$$

Via the substitution $z = e^{-t}$, the function $D_1(e^{-t})$ falls under the harmonic sum so that its Mellin transform is

$$\mathcal{M}[D_1(e^{-t}); s] = \zeta(s)\mathcal{M}[\alpha(e^{-t}); s].$$

The asymptotic expansion

$$\alpha(e^{-t}) = \frac{1}{2t} - \frac{\sqrt{2}}{6}\frac{1}{\sqrt{t}} - \frac{1}{18} + O(\sqrt{t}),$$

gives the singular expansion of the corresponding Mellin transform. This in turn yields the singular expansion of $\mathcal{M}[D_1(e^{-t}); s]$. Then, the reverse mapping Theorem 9.2 gives back $D(e^{-t})$ at $t \sim 0$, hence,

$$D_1(z) = \frac{1}{2}\frac{1}{1-z}\log\frac{1}{1-z} + \frac{1}{2}\frac{\gamma}{1-z} - \frac{1}{6}\frac{\sqrt{2}\zeta(\frac{1}{2})}{\sqrt{1-z}} - \frac{1}{4}\log\frac{1}{1-z} + O(1),$$

where $\gamma = 0.577\ldots$ is the Euler constant. The combination of this last estimate and the main asymptotic form of $S(z,1)$ in (9.72) yields

$$S'_u(z,1) \sim \frac{1}{2}a\exp\left(\frac{c}{1-z} + \frac{3}{4}\log\frac{1}{1-z} + \log\log\frac{1}{1-z}\right),$$

where a is the same constant as in (9.72). Like for $S(z,1)$, the derivative $S'_u(z,1)$ is amenable to Lemma 9.9, and this proves the asymptotic form of μ_n, as stated in (9.61) of Lemma 9.7.

Finally, we need to justify (9.62) that represents a bound on the variance of K_n. The computations follow the same steps as above, so we only sketch them briefly. One needs to estimate a second derivative,

$$\frac{S''_{uu}(z,1)}{S(z,1)} = D_2(z) + D_1^2(z)$$

where

$$D_2(z) = \sum_{k=0}^{\infty} z^{2k}\frac{\beta''(z^k)}{\beta(z^k)} - \left(\frac{z^k\beta'(z^k)}{\beta(z^k)}\right)^2.$$

The sum above is again a harmonic sum that is amenable to Mellin analysis, with the result that

$$D_2(z) = \frac{\zeta(2)}{2}\frac{1}{(1-z)^2} + O((1-z)^{-3/2}).$$

Then we appeal again to Lemma 9.9 to achieve the transfer to coefficients. Somewhat tedious calculations (that were assisted by the computer algebra system MAPLE) show that the leading term in $n\log^2 n$ of the second moment cancels with the square of the mean μ_n. Hence, the variance cannot be larger than $O(n\log n)$. This establishes the second moment estimate (9.62) of Lemma 9.7.

In summary, we proved that the minimax redundancy R_n^* attains the following asymptotics

$$R_n^* = \frac{2}{\log 2}\sqrt{\left(\frac{\pi^2}{6}-1\right)n} - \frac{5}{8}\log_2 n + \frac{1}{2}\log_2\log n + O(1)$$

as $n \to \infty$.

9.5 Extensions and Exercises

9.1 Prove Entry 13 of Table 9.1.

9.2 Prove that

$$\mathcal{M}\left[e^{-x} - \sum_{j=0}^{k}(-1)^k\frac{x^j}{j!}; s\right] = \Gamma(s)$$

for $-k-1 < \Re(s) < -k$. What is the Mellin transform of $e^{-x}-2$, if it exists.

9.3 ⚠ (Louchard, Szpankowski and Tang 1999) Prove the following result.

Lemma 9.11 *Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of real numbers. Suppose that its Poisson generating function $\widetilde{F}(z) = \sum_{n=0}^{\infty} f_n\frac{z^n}{n!}e^{-z}$ is an entire function. Furthermore, let its Mellin transform $F(s)$ have the following factorization: $F(s) = \mathcal{M}[\widetilde{F}(z); s] = \Gamma(s)\gamma(s)$, and assume $F(s)$ exists for $\Re(s) \in (-2,-1)$ while $\gamma(s)$ is analytic for $\Re(s) \in (-\infty,-1)$. Then*

$$\gamma(-n) = \sum_{k=0}^{n}\binom{n}{k}(-1)^k f_k, \quad \text{for} \quad n \geq 2.$$

9.4 ⚠ Define

$$\begin{aligned} f(x) &= \sum_{k\geq 1} \frac{e^{kx}}{\left(e^{kx}-1\right)^2} \\ g(x) &= \sum_{k\geq 1} \frac{(-1)^{k-1}}{k\left(e^{kx}-1\right)}, \\ h(x) &= \sum_{k\geq 1} \frac{1}{k\left(e^{2k\pi x}-1\right)}. \end{aligned}$$

Prove the following identities:

$$\begin{aligned} 1+2\sum_{n\geq 1} e^{-\pi n^2 x} &= \frac{1}{\sqrt{x}} + \frac{2}{\sqrt{x}} \sum_{n\geq 1} e^{-\pi n^2/x} && \text{Jacobi's } \vartheta\text{–function,} \\ f(x) &= \frac{1}{x^2}\frac{\pi^2}{6} - \frac{1}{2x} + \frac{1}{24} - \frac{4\pi^2}{x^2} f\left(\frac{4\pi^2}{x}\right) && \text{Ramanujan [35],} \\ g(x) &= \frac{\pi^2}{12x} - \frac{\log 2}{2} + \frac{x}{24} - g\left(\frac{2\pi^2}{x}\right) && \text{Ramanujan [35]} \\ h(x) &= \frac{\pi}{12}\left(\frac{1}{x} - x\right) + \frac{\log x}{2} + h\left(\frac{1}{x}\right) && \text{Dedekind and Ramanujan [35].} \end{aligned}$$

The latter three identities play a role in the analysis of the variance for some parameters of tries (cf. [199, 206, 207, 209]).

9.5 Consider the following functional equation

$$f(x) = 2f(x/2) + 1 - (1+x)e^{-x}, \qquad x > 0.$$

Prove that for arbitrary positive M and $x \to \infty$

$$f(x) = x\left(\frac{1}{\log 2} + G(\log_2 x)\right) - 1 + O(x^{-M})$$

where

$$G(x) = \frac{1}{\log^2 2} \sum_{k\neq 0} k\Gamma(-1 + 2\pi ik/\log 2)\exp(-2\pi ikx)$$

is a periodic function with period 1 and amplitude smaller than $1.6 \cdot 10^{-6}$.

9.6 Using the generalized Euler-Maclaurin formula discussed in Example 4, analyze asymptotically

$$F_1(x) = \sum_{n\geq 1} e^{-\sqrt{nx}},$$

$$F_2(x) = \sum_{n\geq 1} \frac{\sqrt{nx}}{1+n^2x^2}.$$

9.7 Prove that under suitable condition (state them!)

$$\sum_{n=1}^{\infty}(-1)^{n-1}f(nx) \sim \sum_{k=0}^{\infty} c_k(1-2^{1+\beta_k})\zeta(-\beta_k)x^{\beta_k}$$

where like in Example 4 we assume that

$$f(x) = \sum_{k=0}^{\infty} c_k x^{\beta_k}, \qquad x \to 0.$$

where $-1 < \beta_0 < \beta_1 < \cdots$.

9.8 Generalize formula (9.37), that is, find asymptotic expansion for

$$F_m(x) = \sum_{n\geq 1} \log^m n f(nx)$$

for $m \geq 1$.

9.9 Develop asymptotic formula for

$$\sum_{n=1}^{\infty}(-1)^n f(nx)\log n.$$

Using this formula establish an asymptotic expansion of the sum from the introduction, namely

$$H(x) = \sum_{k=1}^{\infty}(-1)^k e^{-k^2 x}\log k$$

for $x \to 0$.

9.10 Consider the function $\varphi(z)$ defined in Example 2, that is,

$$\varphi(z) = \prod_{j=0}^{\infty}\left(1 - e_d(z2^j)e^{-z2^j}\right)$$

where $e_d(z) = 1 + \frac{z^1}{1!} + \cdots + \frac{z^d}{d!}$. Prove that for all $z \in \mathcal{S}_\theta$ with $z \neq 0$ and $|\theta| < \pi/2$ we have uniformly $|\varphi(z)| < A$ for some constant $A > 0$.

9.11 Prove that

$$F(x) = \sum_{k=1}^{\infty} \left(1 - (1 - 2^{-k})^x\right)$$

attains the following asymptotics for $x \to \infty$

$$F(x) \sim \log_2 x + \frac{\gamma}{\log 2} - \frac{1}{2} + \sum_{j=1}^{\infty} P_j(\log_2 x) x^{-j}$$

where P_j is a periodic function of period 1.

9.12 Analyze asymptotically

$$\sum_{k \geq 0} (-1)^k \sqrt{k} \binom{2n}{n-k}.$$

9.13 ⚠ Let P_n be the number of integer partitions. In Example 5 we proved that its ordinary generating function is

$$P(z) = \prod_{k=1}^{\infty} (1 - z^k)^{-1}.$$

Show that

$$P_n \sim \frac{\exp(\pi\sqrt{2n/3})}{4\sqrt{3}n}.$$

Hint. The following steps are advised:

1. Define

$$F(t) = \log P(e^{-t}) = \sum_{k=1}^{\infty} \log(1 - e^{kt})^{-1}$$

and then prove that the Mellin transform $F^*(s)$ of $F(t)$ is

$$F^*(s) = \zeta(s)\zeta(s+1)\Gamma(s).$$

2. Using the reverse mapping Theorem 9.2 prove that

$$F(t) \sim \frac{\pi^2}{6t} + \log\sqrt{\frac{t}{2\pi}} - \frac{t}{24}, \qquad t \to 0$$

which yields

$$P(z) = \frac{e^{-\pi^2/12}}{\sqrt{2\pi(1-z)}} \exp\left(\frac{\pi^2}{6(1-z)}\right)(1 + O(1-z)).$$

3. Use the saddle point method (as described in Table 8.4 or in Lemma 9.9) to establish the result.

9.14 ⚠ Let Q_n be the number of integer partitions in distinct parts. Prove that its generating function is

$$Q(z) = \prod_{k=1}^{\infty}(1+z^k)$$

and then establish

$$Q_n \sim \frac{\exp(\pi\sqrt{n/3})}{43^{1/4}n^{3/4}}.$$

9.15 ⚠ (Andrews 1984) Let

$$G(z) = \prod_{n=1}^{\infty}(1-z^n)^{-a_n} = 1 + \sum_{n=1}^{\infty} r_n z^n$$

where $a_n \geq 0$ such that its the Dirichlet series $D(s) = \sum_{n\geq 1}\frac{a_n}{n^s}$ is convergent for $\Re(s) > \alpha$ and

$$D(s) \asymp \left[\frac{A}{s-\alpha}\right]_{s=\alpha}.$$

Furthermore, assume that $D(s)$ is continuable to the left and of moderate growth, that is, $D(s) = O(|\Im(s)|^c)$ for some constant c. Let $z = e^{-t}$ and we assume that for $\arg(t) > \pi/4$.

$$\Re(G(e^{-t})) - G(e^{-\Re(t)}) \leq -C_2\Re(t)^{-\varepsilon}$$

for sufficiently small $\Re(t)$ and arbitrary small $\varepsilon > 0$. Generalize the approach presented in Exercise 13 to prove the following lemma due to G. Meinardus (cf. [12]).

Lemma 9.12 (Meinardus) *With the notations as above, the following holds*

$$r_n \sim Cn^{\kappa}\exp\left(n^{\alpha/(1+\alpha)}(1+1/\alpha)\left(A\Gamma(\alpha+1)\zeta(\alpha+1)\right)^{1/(1+\alpha)}\right)$$

where

$$\begin{aligned} C &= \frac{e^{D'(0)}\left(A\Gamma(\alpha+1)\zeta(\alpha+1)\right)^{(1-2D(0))/(2+2\alpha)}}{\sqrt{2\pi(1+\alpha)}} \\ \kappa &= \frac{D(0)-1-\alpha/2}{1+\alpha} \end{aligned}$$

as $n \to \infty$.